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A
Appendix

A.1
Some ergodic theory

We recall the basic tools from ergodic theory used in the text. For the proofs of these statements, we refer the reader to (9) and (8).

A system from the point of view of ergodic theory $(\Omega, \mathcal{B}, \mu, T)$ is composed by a set Ω, a σ-algebra $\mathcal{B}$ over Ω, a probability measure μ on $\mathcal{B}$ and a measurable map $T : \Omega \to \Omega$ that preserves μ, i.e., a map T such that $\mu(T^{-1}A) = \mu(A)$, for every $A \in \mathcal{B}$.

The first simple remark to be made is the following theorem.

Theorem A.1.1 (Poincaré's recurrence theorem)**.** *If Ω is a second-countable Hausdorff space and $\mathcal{B}$ the Borel σ-algebra, then μ-almost every point is recurrent, i.e., the set*

$$\{x \in \Omega \ : \ x \text{ is a limit point of } (T^n(x))_{n \geq 0}\}$$

is of full μ-measure.

A system is said to be *ergodic* if every measurable set A satisfying $T^{-1}A = A$ also satisfies $\mu(A) = 0$ or 1. Equivalently, a system is ergodic if every integrable function $\phi : \Omega \to \mathbb{R}$ satisfying $\phi \circ T = \phi$ is constant μ-almost everywhere. Ergodicity is the notion of indecomposability of a system. We may also say "T is ergodic (with respect to μ)", "μ is ergodic (with respect to T)", etc.

The fundamental theorem of ergodic theory is the following.

Theorem A.1.2 (Birkhoff's ergodic theorem)**.** *Let* $(\Omega, \mathcal{B}, \mu, T)$ *be a system and* $\phi : \Omega \to \mathbb{R}$ *an integrable function. Then, the limit*

$$\tilde{\phi}(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(T^i(\omega))$$

exists μ*-a.e.. The function* $\tilde{\phi}$ *is integrable,* T*-invariant and satisfies*

$$\int_\Omega \tilde{\phi}\ d\mu = \int_\Omega \phi\ d\mu.$$

Most books on ergodic theory have a proof of Birkhoff's theorem. For a classic proof, the references given suffice. For a modern, extremely simple proof, see (1). The following theorem is a far-reaching generalisation of Birkhoff's theorem.

Theorem A.1.3 (Subadditive ergodic theorem)**.** *Let* $(\Omega, \mathcal{B}, \mu, T)$ *be a system and* $W_i : \Omega \to \mathbb{R}$ *a sequence of positive measurable functions such that* W_1 *is integrable and*

$$W_{n+k} \leq W_k + W_n \circ T^k$$

for all positive integers n, k*. Then the limit*

$$W(\omega) = \frac{1}{n} W_n(\omega)$$

exists μ*-a.e., is* T*-invariant and integrable.*

Remark A.1.4. *If the systems in A.1.2 and A.1.3 are ergodic, then the respective limits are constant* μ*-a.e..*

An important special case of Birkhoff's theorem for ergodic systems is the case of the caracteristic function of a measurable subset A. In this case, the theorem tells us that μ-a.e. point of Ω visits A with frequency $\mu(A)$. We can often suppose that a system is ergodic, since every system admits a standard decomposition in ergodic systems, using the following theorem.

Theorem A.1.5 (Ergodic decomposition theorem)**.** *Let* $(\Omega, \mathcal{B}, \mu, T)$ *be a system, where* Ω *is a compact metric space and* $\mathcal{B}$ *the Borel* σ*-algebra. Denote by* Λ *the convex set of probabilities on* $\mathcal{B}$ *which are* T*-invariant, seen as a subset of the vector space of finite signed measures on* $\mathcal{B}$*. There exists a probability* Θ *on* Λ*, supported on the ergodic probabilities such that*

$$\mu = \int_\Lambda \nu\ d\Theta(\nu).$$

This theorem may be proved directly (see (9)) or seen as a corollary of Choquet's theorem. Finally, we have the following existence theorem.

Theorem A.1.6. *If Ω is a compact metric space and $T : \Omega \to \Omega$ is continuous, then there exists a T-invariant probability on the Borel σ-algebra over Ω.*

A.2
Other theorems cited in the text

A.2.1
Lefschetz's fixed point theorem

Consider a compact manifold X and a continuous map $f : X \to X$. This induces, by simple composition, an endomorphism $f_{\#} : C_n(X) \to C_n(X)$ of the group of singular n-chains (with coefficients in $\mathbb{Z}$), for every integer n. By verifying that $f_{\#}$ commutes with the boundary operator ∂, we conclude that $f_{\#}$ induces homomorphism $f_* : H_n(X) \to H_n(X)$ of the homology groups. This depends only on the homotopy class of f. The groups $H_n(X)$ are abelian and finitely generated, admitting a canonical decomposition in torsion and torsion-free parts. Choosing a basis for the torsion-free part, the homomorphism f_* acts on the torsion-free part of $H_n(X)$ as multiplication by an integer matrix $[a_{ij}]$. The trace of f_* will be defined as the trace of this matrix and it does not depend on the chosen basis. The *Lefschetz number of f* is

$$\tau(f) = \sum_n (-1)^n \mathrm{tr}(f_* : H_n(X) \to H_n(X)).$$

Theorem A.2.1 (Lefschetz's fixed point theorem)**.** *If $\tau(f) \neq 0$, then f has a fixed point.*

A proof may be found in (7). If f is homotopic to the identity, we have

$$\begin{aligned} \tau(f) = \tau(id) &= \sum_n (-1)^n \mathrm{tr}(Id : H_n(X) \to H_n(X)) \\ &= \sum_n (-1)^n \text{ rang } H_n(X) = \chi(X), \end{aligned}$$

where $\chi(X)$ is the Euler caracteristic of X and the last equality is a simple theorem of homological algebra. We arrive to the following corollary.

Corollary A.2.2. *If f is homotopic to the identity and X is a closed surface other than the torus $\mathbb{T}^2$, then f has a fixed point.*

A.2.2
Brouwer's translation arc theorem

Brouwer's theory is a collection of results on the dynamics of the orientation-preserving homeomorphisms $h : \mathbb{R}^2 \to \mathbb{R}^2$. Very generally, it says that any form of recurrence for these maps implies the existence of fixed points. The central theorem, Brouwer's translation arc theorem, says that a fixed-point free orienation-preserving homeomorphism of the plane can be obtained by "gluing" plane translations. We will only use a weaker corollary of Brouwer's theory and, by an abuse, we shall also call it translation arc theorem.

Theorem A.2.3 (Brouwer's translation arc theorem)**.** *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-preserving homeomorphism. If h has a periodic point, then h has a fixed point.*

A modern treatment to Brouwer's theory can be found in (2).