6
Quasi-isometries

6.1
Elements from geometric group theory

Let Γ be a finitely generated group and $\mathcal{S}$ a *finite* generating set.

Definition 6.1.1. *The word norm of an element $\gamma \in \Gamma$ with respect to $\mathcal{S}$ is the length of the shortest word in $\mathcal{S} \cup \mathcal{S}^{-1}$ that represents γ in Γ. It will be denoted by $|\gamma|_{\mathcal{S}}$ or simply $|\gamma|$. The norm of the identity element is defined to be zero and the word distance between two elements $\gamma_1, \gamma_2 \in \Gamma$ is $|\gamma_1^{-1}\gamma_2|$. Equivalently, $|\gamma|$ is the "shortest path" distance between γ and the identity in the Cayley graph of Γ with respect to $\mathcal{S}$ (each edge having unit length).*

This notion turns the group Γ into a metric space. While different choices of a generating set yield different metrics, the following definition and proposition show how geometric group theory deals with this ambiguity.

Definition 6.1.2. *Two metric spaces (X, d) and (X', d') are said to be* quasi-isometric *if there exists a pair of maps $f : X \to X'$ and $g : X' \to X$ and a pair of constants $\lambda \geq 1$ and $C \geq 0$ satisfying the following conditions:*

1. $d'(f(x), f(y)) \leq \lambda d(x, y) + C$,
2. $d(g(x'), g(y')) \leq \lambda d'(x', y') + C$,
3. $d(g(f(x)), x) \leq C$ *and* $d'(f(g(x')), x') \leq C$,

for every $x, y \in X$ and $x', y' \in X'$. The maps f and g are called (λ, C)-quasi-isometries.

Proposition 6.1.3. *For a finitely generated group, all word metrics are quasi-isometric.*

Proof. If $\mathcal{S}$ and $\mathcal{T}$ are two finite generating sets, we take

$$\lambda = \max\left(\max_{\gamma \in \mathcal{S}} |\gamma|_{\mathcal{T}}, \max_{\gamma \in \mathcal{T}} |\gamma|_{\mathcal{S}}\right)$$

and $C = 0$. $\square$

6.2
A quasi-isometry lemma for surfaces of finite type

It is a classical result that the fundamental group of a compact surface is quasi-isometric to the hyperbolic disk (see (6)). In this section, we shall adapt the proof of this lemma to encompass open surfaces of finite type. We begin by some concepts and facts from hyperbolic manifolds. We will omit the proofs and refer the reader to (12).

We recall that isometries of $\mathbb{D}$ admit natural extensions to homeomorphisms of the closed disc $\overline{\mathbb{D}}$. We fix a point x_0 in $\mathbb{D}$. The set of limit points of Γx_0 in the boundary circle $\partial\mathbb{D}$ (for the standard topology on $\mathbb{D}$) does not depend on x_0: for if y_0 is another point of $\mathbb{D}$ and p is the limit of a sequence $\sigma_i x_0$, then the hyperbolic distance $d(\sigma_i x_0, \sigma_i y_0)$ is equal to $d(x_0, y_0)$ for every integer i, since Γ acts by isometries. Since $\sigma_i x_0$ tends to the boundary circle, the euclidian distace between $\sigma_i x_0$ and $\sigma_i y_0$ tends to zero and, consequently, $\sigma_i y_0$ tends to p as well.

Definition 6.2.1. *The set $L(\Gamma)$ of limit points of Γx_0 in $\partial\mathbb{D}$ is called* limit set of Γ. *A subset R of $\overline{\mathbb{D}}$ is said to be* (hyperbolically) convex *if, for every pair of points in R, the geodesic segment joining them lies in R as well. The* (hyperbolic) convex hull *of a subset R of $\overline{\mathbb{D}}$ is the smallest convex subset of $\overline{\mathbb{D}}$ containing R. The* convex hull *of $L(\Gamma)$ will be denoted by $C(\Gamma)$.*

Since $L(\Gamma)$ is closed and invariant in $\partial\mathbb{D}$, the set $C(\Gamma) \cap \mathbb{D}$ is closed and invariant in $\mathbb{D}$.

Definition 6.2.2. *The quotient $C(S) = (C(\Gamma) \cap \mathbb{D})/\Gamma$ is called* convex core *of S.*

It is known that, if S is of finite type, we can cut $C(S)$ along a finite number of closed geodesics in order to obtain a compact convex subset K. Since the boundary of K is geodesic and two different geodesics of $\mathbb{D}$ cannot intersect more than once, an arc of geodesic starting in K which leaves K must remain outside K.

We are now ready to prove the following lemma.

Lemma 6.2.3. *With g defined as in Section 5.1, the spaces ΓK_0 and Γ, endowed respectively with the hyperbolic metric (inherited from $\mathbb{D}$) and with a word metric are quasi-isometric.*

Proof. We denote by d the hyperbolic metric on $\mathbb{D}$ and d_Γ the word metric on Γ, with respect to a fixed set of generators. Choose $x_0 \in K_0$ and define

$$\begin{aligned} f:\Gamma &\rightarrow \Gamma K_0 \\ \gamma &\mapsto \gamma x_0. \end{aligned}$$

Each point $x \in \Gamma K_0$ is in exactly one region of the form γK_0. Thus we obtain a map $g : \Gamma K_0 \to \Gamma$. We verify that these maps give the desired quasi-isometry: $g \circ f$ is the identity and $f \circ g$ does not move points more than the hyperbolic diameter of K_0, which is finite. This shows that these maps satisfy condition 3.

To prove 2, set R to be the hyperbolic diameter of K_0, B the closed ball of radius R centered on x_0,

$$C = \{\gamma \in \Gamma : \gamma \neq \text{id and } \gamma B \cap B \neq \emptyset\} \quad \text{and} \quad r = \inf\{d(B, \gamma B) : \gamma \in \Gamma - (C \cup \{id\})\}.$$

Note that C is finite, since the action of a group of deck transformations is always properly discontinuous. We claim that r is strictly positive. Indeed, the set $E = \{\gamma \in \Gamma : d(x_0, \gamma x_0) \leq 4R\}$ is finite (the action of Γ is properly discontinuous). Hence, $r' = \inf\{d(B, \gamma B) : \gamma \in E\}$ is not zero. It then follows that $r \geq \min(r', R) > 0$, as claimed.

Now consider an arbitrary $\gamma \in \Gamma$. Let k be the smallest integer satisfying $d(x_0, \gamma x_0) < kr + R$ and choose points $x_1, \ldots, x_{k+1} = \gamma x_0$ on the geodesic segment $[x_0, \gamma x_0]$ such that $d(x_0, x_1) \leq R$ and $d(x_i, x_{i+1}) \leq r$ for $1 \leq i \leq p$. Since the geodesic segment is contained in ΓK_0, one may choose $\gamma_1, \ldots, \gamma_{k+1} \in \Gamma$ such that $x_i \in \gamma_i B$. Setting $\sigma_i = \gamma_i^{-1}\gamma_{i+1}$, we have $\gamma = \sigma_i \ldots \sigma_k$,

$$\gamma_i^{-1} x_i \in B, \qquad \gamma_i^{-1} x_{i+1} = \sigma_i \gamma_{i+1}^{-1} x_{i+1} \in \sigma_i B \qquad \text{and} \qquad d(\gamma_i^{-1} x_i, \gamma_i^{-1} x_{i+1}) < r.$$

By the definition of r, this implies that $\sigma_i \in C \cup \{id\}$. In particular, C generates Γ and, with respect to C, we have $d_\Gamma(1, \gamma) \leq k$. It follows from the minimality of k that

$$(k-1)r + R \leq d(x_0, \gamma x_0) \qquad \text{and} \qquad d_\Gamma(1, \gamma) \leq k \leq \frac{1}{r} d(x_0, \gamma x_0) + 1.$$

Finally, it remains to check that condition 1 is verified. Set $L = \max_{\sigma \in C} d(x_0, \sigma x_0)$. Now, for every pair $\gamma_1, \gamma_2 \in \Gamma$, we can write $\gamma_2 \gamma_1^{-1} =$

$\sigma_{i_1} \dots \sigma_{i_k}$, therefore obtaining

$$\begin{aligned} d(f(\gamma_1), f(\gamma_2)) &= d(\gamma_1 x_0, \gamma_2 x_0) \\ &\leq d(\gamma_1 x_0, \sigma_{i_k}\gamma_1 x_0) + \dots + d(\sigma_{i_2} \dots \sigma_{i_k}\gamma_1 x_0, \sigma_{i_1} \dots \sigma_{i_k}\gamma_1 x_0) \\ &= d(x_0, \sigma_{i_k} x_0) + \dots + d(x_0, \sigma_{i_1} x_0) \\ &\leq Lk = L\ d_\Gamma(\gamma_1, \gamma_2). \end{aligned}$$

□

Note that, in this proof, K may be replaced by any other compact containing it. It suffices to chosse x_0 in K and carry out the proof by verbatim.

6.3 Proof of the main result

Combining the previous lemma and the positive linear growth already proved (Lemma 5.2.3), we arrive at the main result.

Theorem 6.3.1. *The sequence $F^n(x)$ of iterates of a μ-generic point x in $\mathbb{D}$ tends to the boundary of $\mathbb{D}$ with positive speed. In other words, if d is the hyperbolic metric, there exists a constant $\bar{m} > 0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} d(x, F^n(x)) = \bar{m}$$

for μ-almost every x in $\mathbb{D}$.

Proof. The almost-everywhere existence of the limit is yet another application of the subadditive ergodic theorem, to the system $f : S \to S$ and to the functions $W_i = d(x, F^i(x))$. Note that W_1 is integrable because the support of the measure μ is compact. The limit is constant due to its f-invariance and to the ergodicity of f. Now we show that $m > 0$ implies $\bar{m} > 0$. Choose $K \subset S$ a compact set of positive measure that contains the convex core of S. By Birkhoff's theorem, a μ-generic point x in S visits K infinitely many times. Let n_i be the set of indices such that $F^{n_i}(x)$ belongs to ΓK (here S is identified with S_0). By Lemma 6.2.3,

$$\frac{1}{n_i}|g(F^{n_i}(x))| \leq \frac{1}{n_i}\left(\lambda d(x, F^{n_i}(x)) + C\right).$$

Hence, $\bar{m} \geq m/\lambda > 0$. □