4
Rotation vectors

In this section we want to define an analogue of the rotation number of orientation-preserving homeomorphisms of the circle. We recall that for one such homeomorphism f, having a lift $F : \mathbb{R} \to \mathbb{R}$, the rotation number is defined as

$$\lim_{n\to\infty} \frac{F^n(x) - x}{n}.$$

One shows that this limit exists, does not depend on the point x chosen and that its value mod 1 does not depend on the choice of the lift F.

4.1
Rotation vectors of homeomorphisms of the torus $\mathbb{T}^2$

There is a straightforward generalisation of the rotation number for homeomorphisms of $\mathbb{T}^2$ isotopic to the identity. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism isotopic to the identity and μ a probability that is invariant by f. If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of f, we set

$$v_\mu(F) = \int_\Omega (F(x) - x)\ d\mu,$$

where Ω is a fundamental region for the covering and μ also represents the pullback of the original measure to $\mathbb{R}^2$. Since F is also isotopic to the identity (by lifting the isotopy between f and the identity), $v_\mu(F)$ does not depend on the choice of Ω.

We set $G(x) = F(x) - x$ and

$$\tilde{G}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} G \circ F^i(x) = \lim_{n\to\infty} \frac{F^n(x) - x}{n},$$

whose μ-almost everywhere (a.e.) existence is given by Birkhoff's theorem (Theorem A.1.2). The same theorem shows that

$$\int_\Omega \left(\lim_{n\to\infty} \frac{F^n(x) - x}{n} \right)\ d\mu = \int_\Omega \tilde{G}(x)\ d\mu = \int_\Omega G(x)\ d\mu = v_\mu(F).$$

This encourages us to take $v_\mu(F)$ as the *mean rotation vector of F (with respect to μ)*. If, in addition, f is ergodic with respect to μ, the function $\tilde{G}$ is constant μ-a.e. and its value is $v_\mu(F)$.

Since all lifts of f commute with the deck transformations of $\mathbb{R}^2 \to \mathbb{T}^2$, the value of $v_\mu(F)$ mod $\mathbb{Z}^2$ does not depend on F. Hence, we define the *mean rotation vector of f with respect to μ, $v_\mu(f)$,* as $v_\mu(F)$ mod $\mathbb{Z}^2$, where F is any lift of f. A simple change of coordinates proves the following lemma.

Lemma 4.1.1. *The map v_μ is a homomorphism between the group of homeomorphisms of $\mathbb{T}^2$ isotopic to the identity that preserve μ, denoted by $\mathrm{Diff}_\mu(\mathbb{T}^2)_0$, and $\mathbb{T}^2$.*

Proof. If f_1 and f_2 are in $\mathrm{Diff}_\mu(\mathbb{T}^2)_0$ and F_1 and F_2 are respective lifts, we have

$$\begin{aligned} v_\mu(F_1 \circ F_2) &= \int_\Omega (F_1(F_2(x)) - x)\, d\mu \\ &= \int_\Omega (F_1(F_2(x)) - F_2(x))\, d\mu + \int_\Omega (F_2(x) - x)\, d\mu \\ &= \int_{F_2(\Omega)} (F_1(x) - x)\, d\mu + \int_\Omega (F_2(x) - x)\, d\mu \\ &= v_\mu(F_1) + v_\mu(F_2), \end{aligned}$$

since F_2 preserves μ and $F_2(\Omega)$ is also a fundamental region. Taking both members of the equation mod $\mathbb{Z}^2$ proves the lemma. $\square$

This lemma show in particular that if an element f in $\mathrm{Diff}_\mu(\mathbb{T}^2)_0$ can be written as the commutator of two elements of $\mathrm{Diff}_\mu(\mathbb{T}^2)_0$, then its rotation vector is zero. The following theorem was proved by J. Franks in (4).

Theorem 4.1.2. *If the rotation vector of $f \in \mathit{Diff}_\mu(\mathbb{T}^2)_0$ is zero and f is ergodic with respect to μ, then f has a fixed point.*

Finally, we show how to compute $\tilde{G}(x)$. This will be useful to generalise the rotation vector in the following section. Fix a fundamental region Ω of bounded diameter (e.g. the unit square) and a point x in Ω. For each n, there exists an unique deck transformation $\delta_n(x)$ such that $F^n(x)$ lies in $\delta_n(x)\Omega$. Since the diameter of Ω is bounded, the distance between $F^n(x) - x$ and $\delta_n(x)$ is bounded. Hence,

$$\tilde{G}(x) = \lim_{n\to\infty} \frac{\delta_n(x)}{n}.$$

4.2
Rotation vectors of homeomorphisms of a surface of finite type

Now, let $f : S \to S$ be an homeomorphism of a surface of finite type (i.e., a surface with finitely generated fundamental group), isotopic to the identity and which preserves a probability measure μ. We define H as the commutator subgroup $[\pi_1(S), \pi_1(S)]$. The subgroup H is normal and the quotient $\pi_1^{ab}(S) = \pi_1(S)/H$ is abelian and finitely generated. There exists a corresponding normal covering $S^{ab} \to S$ whose group of deck transformations is isomorphic to $\pi_1^{ab}(S)$. Since f is isotopic to the identity, it can be lifted to an homeomorphism $F : S^{ab} \to S^{ab}$. As in the preceding section, given a fundamental region Ω and a point x in Ω, we denote by $\delta_n(x)$ the deck transformation such that $F^n(x)$ lies in $\delta_n(x)\Omega$. We set

$$\tilde{G}(x) = \lim_{n\to\infty} \frac{\delta_n(x)}{n} \in \pi_1^{ab}(S) \otimes \mathbb{R}$$

and

$$v_\mu(F) = \int_\Omega \tilde{G}\, d\mu.$$

The μ-a.e. existence of this limit is also a consequence of Birkhoff's theorem.

Remark 4.2.1. *The group $\pi_1^{ab}(S)$ is canonically isomorphic to the first homology group with integer coefficients $H_1(S,\mathbb{Z})$. Abelian groups have a natural structure of module over $\mathbb{Z}$ and the tensor product $\otimes$ is meant in this sense. The real vector space $\pi_1^{ab}(S) \otimes \mathbb{R}$ is isomorphic to the first real homology $H_1(S,\mathbb{R})$. These isomorphisms come from the Hurewicz theorem and the universal coefficient theorem for homology (see (7)). This will not be necessary in the sequel.*

Lifting the isotopy between f and the identity, we have a lift $F : S^{ab} \to S^{ab}$ that commutes with all deck transformations: let H_t be the lifted isotopy. If σ is a deck transformation, then $\sigma H_t \sigma^{-1} H_t^{-1}$, being the difference of two lifts of the same map h_t, is a deck transformation ρ_t. Since the map $t \mapsto \rho_t$ is continue and the deck transformation group is discrete, it is constant equal to ρ_0, which is the identity. The lift F is called an *identity lift*.

The difference of two identity lifts is a deck transformation that commutes with all other deck transformations. Since the fundamental group of a surface is either abelian or has trivial center, we have that either every lift of f is an identity lift, or that the identity lift is unique. In the first case, we set $v_\mu(f)$ as the image of $v_\mu(F)$ in the quotient of $\pi_1^{ab}(S) \otimes \mathbb{R}$ by the torsion-free part of $\pi_1^{ab}(S)$, since this does not depend on the choice of the lift F. For the latter case, we take $v_\mu(f)$ to be $v_\mu(F)$, where F is the only identity lift. When f is ergodic with respect to μ, $v_\mu(f)$ will be called simply *rotation vector of* f.