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2 General Setting

Let M be a closed connected orientable surface and $f : M \to M$ a C^1 diffeomorphism. The set of fixed points of f will be denoted by $\mathrm{Fix}(f)$.

Definition 2.0.4. *The diffeomorphism $f : M \to M$ is said to be* isotopic to the identity *if there exists a continuous map*

$$\begin{array}{rcl} h : [0,1] \times M & \to & M \\ (t,x) & \mapsto & h_t(x) \end{array}$$

such that $h_0 : M \to M$ is the identity map, h_1 coincides with f and h_t is a diffeomorphism for every t in $[0,1]$. If we can find such an h that, in addition, the diffeomorphisms h_t pointwisely fix $\mathrm{Fix}(f)$, *then f is said to be* isotopic to the identity relatively to $\mathrm{Fix}(f)$.

If f is isotopic to the identity and M is not the torus, then f has a fixed point, by the Lefschetz fixed point theorem (Theorem A.2.1). For the torus, the supplementary hypothesis of "vanishing rotation vector", which will be presented in Chapter 4, also assures the existence of a fixed point. We shall thus assume that $\mathrm{Fix}(f)$ is not empty and that f is isotopic to the identity relatively to $\mathrm{Fix}(f)$.

Suppose that f preserves a probability measure μ, that we may suppose ergodic by the ergodic decomposition theorem (Theorem A.1.5). Since $\mathrm{Fix}(f)$ is f-invariant, we have $\mu(\mathrm{Fix}(f)) = 0$ or 1. We shall assume that $\mu(\mathrm{Fix}(f)) = 0$.

Since $\mathrm{Fix}(f)$ is not empty, $S = M \setminus \mathrm{Fix}(f)$ is an open surface. The restriction $f : S \to S$ permutes the connected components of S and is isotopic to the identity. We conclude that a component of positive measure must be periodic, since the total measure is finite. By ergodicity, there exists only one orbit of components of positive measure. Thus, by replacing f by a suitable power f^k and by normalising μ, we may assume without loss of generality that S is connected.

We shall assume, to simplify the exposition, that $\mathrm{Fix}(f)$ is discrete (finite) and that the support of the probability μ, $\mathrm{supp}(\mu)$, is compact in S. In other words, we will suppose that each fixed point of f in M has a neighborhood of zero measure.

Remark 2.0.5. *This last hypothesis is strictly stronger than $\mu(\mathrm{Fix}(f)) = 0$. In fact, it is equivalent to the existence of a f-invariant compact subset of S.*

Proof. One direction is obvious since the support of μ is f-invariant. And if there exists a compact f-invariant subset of S, Theorem A.1.6 assures us of the existence of an invariant probability supported in it. Its support is thus compact in S. □

The surface M may be thought as a polygon whose edges were suitably identified. We obtain S by removing a finite number of points from the interior of this polygon. With this representation, it is easier to see that S retracts to a finite graph. In particular, this shows that the fundamental group $\pi_1(S)$ is free and finitely generated. By the Riemann uniformization theorem, the universal covering of S may be identified with the hyperbolic disk $\mathbb{D}$. We shall choose an ideal hyperbolic polygon S_0 as fundamental region for this covering. When no confusion is possible, we will implicitly identify S and S_0. In other cases, if x is a point of S, we will denote by x_0 the corresponding point in S_0. The same convention will be used for lifting sets from S to S_0.

Remark 2.0.6. *By fundamental region we mean a choice of an unique point from each fiber of the covering. Hence, S_0 does not contain all of its edges. This implies that the lift of a compact from S to S_0 may not be compact. However, lifting a compact subset of S always yields a bounded subset of S_0.*

We will denote by Γ both the fundamental group $\pi_1(S)$ and the group of deck transformations $\mathrm{Aut}_S(\mathbb{D})$, noting that they are canonically isomorphic. Finally, when we lift f to a diffeomorphism $F : \mathbb{D} \to \mathbb{D}$, we will always choose an "identity lift" (see Chapter 4 for a definition).

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