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Introduction

It is known that some classes of diffeomorphisms of closed surfaces only have trivial invariant measures, i.e., those supported on the set of fixed points. For instance, J. Franks and M. Handel proved in (5), using deep techniques from the topological dynamics of surface homeomorphisms (in particular, their classification up to isotopy), the following theorem.

Theorem 1.0.1. *Let M be an orientable closed surface of genus 2 or higher and $f : M \to M$ a C^1 diffeomorphism isotopic to the identity that preserves a probability μ. If f is a distortion element in the group of C^1 diffeomorphisms isotopic to the identity* $\mathrm{Diff}(M)_0$*, then* $\mathrm{supp}(\mu) \subset \mathrm{Fix}(f)$*, where* $\mathrm{Fix}(f)$ *denotes the set of fixed points of f.*

Analogue results are proved for the sphere and the torus. We recall that f is a *distortion element* in $\mathrm{Diff}(M)_0$ (or in any other group G) if f is not a torsion element and if it belongs to a finitely generated subgroup of $\mathrm{Diff}(M)_0$ (resp. G) where

$$\liminf_{n\to\infty} \frac{|f^n|}{n} = 0,$$

$|\cdot|$ denoting the word metric corresponding to such subgroup.

The goal of this dissertation is to show some results obtained this year by the author and his advisor on the following problem, posed by J. Rebelo. Suppose now that $f : M \to M$ preserves a "non-trivial" probabilty measure, i.e., a probability measure whose support is not contained in $\mathrm{Fix}(f)$. What can we say about this diffeomorphism?

Inessential hypothesis were added to make this exposition somewhat lighter and to shed light on the essential ideas. Here is the main result of this text.

Theorem 1.0.2. *Let $S = M \setminus Fix(f)$ and $F : \mathbb{D} \to \mathbb{D}$ be an identity lift of $f|_S$ to the hyperbolic disk obtained by lifting the isotopy between $f|_S$ and the* *identity. Then, there exists a strictly positive constant $\bar{m}$ such that*

$$\lim_{n\to\infty} \frac{d(x, F^n(x))}{n} = \bar{m},$$

for μ-almost every x in $\mathbb{D}$, where d stands for the hyperbolic metric.

This result already yields an interesting corollary for the actions by diffeomorphisms on surfaces, which is also a corollary of (5).

Corollary 1.0.3. *Let G be a finitely generated subgroup of $\mathrm{Diff}(M)_0$ and f a non-periodic central element of G. If f preserves a probability measure whose support is not contained in $\mathrm{Fix}(f)$, then f is not a distortion element in G.*

In particular, as in (5), our result gives a linear estimate for the growth of $|f^n|$, more precise than the estimate $|f^n| \succ \sqrt{n}$ given by L. Polterovich in (11). His work, however, is based on hard techniques of symplectic geometry.

As already mentioned, the techniques in (5) use the classification of surface homeomorphisms in a fundamental way. Hence, there is no hope in applying the same techniques to analogue problems in higher dimension. However, the dichotomy we show between recurrence and positive linear growth in Section 5.3 remains true in dimensions higher than 2. In dimension 3, this plays an important role in the work of C. McMullen (10) and W. Floyd (3).

The idea of our work is to think of the lift $F : \mathbb{D} \to \mathbb{D}$ as a (non-random) walk in a free group and to prove the linear growth in this setting. In analogy with random walks (see Chapter 3), having a free group does not seem necessary. We believe that the same proof may be adapted to encompass the class of amenable groups. Even if the group walk we define is not random, it is defined by an ergodic map, which makes us think that its assymptotic behavior may resemble the behavior of its random analogue.

As in (5), we make use of a result from the Brouwer theory of plane homeomorphisms (see Theorem A.2.3). This result does not have any analogues in higher dimension.