2
Welfare-Improving Debt Constraints

2.1
Introduction

Long-lived assets in positive net supply, such as equity and fiat money, have been widely studied in general equilibrium models. In such models, the relation between dividends and equilibrium prices for those assets depends crucially on whether agents are finitely or infinitely lived. In overlapping generations models, for instance, prices may exceed fundamental values. In models with infinite-lived agents, however, prices tend to coincide with fundamental values – i.e, price bubbles cannot occur (see Magill & Quinzii (1996) and Santos & Woodford (1997)).

The source of the difference is the following. In overlapping generations models, the present value of wealth may be infinite, whereas, in models where agents are infinitely lived, the present value of wealth must be finite whenever the relevant deflators are Kuhn-Tucker multipliers. Indeed, it is for such deflators, and under a condition known as uniform impatience, that assets in positive net supply are not subject to bubbles.

Uniform impatience is a usual requirement for existence of equilibrium in economies with infinite lived debt-constrained agents (see Hernandez & Santos (1996) or Magill & Quinzii (1996)). This condition is satisfied whenever preferences are separable over time and across states so long as (i) the intertemporal discounted factor is constant, (ii) individual endowments are uniformly bounded away from zero, and (iii) aggregate endowments are uniformly bounded from above.

In this paper, we impose a constraint on the amount of debt an agent can have at any given point in time in a model with infinitely lived agents. It is then shown that prices of assets in positive net supply may differ from their expected discounted sum of dividends *even* under uniform impatience. The necessary and sufficient condition needed for this pricing deviation to occur is that each agent must have binding debt constraints now or at some point in the future.

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Throughout the paper, we consider the case of an asset paying no dividends and with positive endowments at the initial node. Samuelson (1958) (see also Tirole (1985)) analyzed such an asset within an overlapping generations context, whereas Bewley (1980), among others (see also Santos & Woodford (1997)), consider this asset in the context of infinite-lived households. Araujo, Páscoa & Novinski (2007) studied this infinite-horizon environment under uncertainty about discounting rates and where, as a consequence, agents become precautious.

The benchmark in which the asset pays no dividends is both simpler and more intriguing. As usual, we call such an asset fiat money.[1] In our model, money may have positive price in equilibrium due to binding debt constraints, which are imposed to avoid Ponzi schemes. This is somehow related to some papers that consider the role of money as a medium of exchange. In Clower (1967), for example, money has positive price in equilibrium because of binding *liquidity constraints*. In a recent work along those lines, Santos (2006) showed that monetary equilibrium can only arise when *cash-in-advance constraints* are binding infinitely often for all agents. In our paper, in contrast, we consider a pure credit economy where money can still be positively valued as a result of agents' desire to take loans when they cannot (either because monetary loans are not allowed or because a debt ceiling has been hit) [2].

As shadow prices of debt constraints play a crucial role in our setting, we develop a duality theory for the households' dynamic programming problem. We identify the Euler and transversality conditions that characterize individual optimality and show that, under the Kuhn-Tucker multipliers, the present value of endowments must be always finite. Once we combine this property with the uniform impatience property, we rule out bubbles in the price of money. Money then can only have positive price if debt constraints are binding. This leads to a somewhat surprising result: credit frictions make room for welfare improvements through intertemporal and inter-states transfers of wealth that are only available when money has positive price.

Our monetary equilibrium is always Pareto inefficient. Indeed, if it were efficient, the agents' rates of intertemporal substitution would necessarily coincide. As money is in positive net supply, at least one agent must carry money, so that his debt constraint is non-binding (and the shadow price is

[1]It is widely recognized that money plays three roles: (i) it is a mean to transfer wealth across time and states (i.e., it stores value), (ii) it is a medium of exchange, and (iii) it is also an unit of account. What we call money in our model plays the role of transferring wealth across time and states. We, therefore, abstract throughout from roles (ii) and (iii) of money.

[2]In a similar context, Gimenez (2005) provided examples of monetary bubbles that can be reinterpreted as positive fundamental values in cashless economies with *no short-sales* restrictions.

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zero). When the agents' rates of intertemporal substitution coincide and the shadow price for one agent is zero, the shadow prices for all agents must be zero, and, as a consequence, the price of money cannot be positive.

When money has a positive value, there exists a deflator, but not one of the Kuhn-Tucker deflators, under which the discounted value of aggregated wealth is infinite and a pure bubble appears. Also, independently of the non-arbitrage deflator, when aggregated endowments can be replicated by a portfolio trading plan, the discounted value of future wealth must be finite (see Santos & Woodford (1997)). Therefore, if we allow for an increasing number of non-redundant securities in order to assure that aggregated wealth can be replicated by the deliveries of a portfolio trading plan, money will have zero price. However, the issue of new assets, in order to achieve that efficacy of the financial markets, can be too costly.

Uniform impatience is key for the results we derive. In fact, we provide an example in which utility functions do not satisfy uniform impatience and speculation in an asset in positive net supply occurs, even for deflators that yield finite present values of wealth.

The rest of the chapter is organized as follows. Section 2 presents the basic model. In Section 3, our results are proved. In the Appendix A we develop the necessary mathematical tools: a duality theory of individual optimization. Other important results are proved in Appendices B and C.

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2.2 Model

We consider an infinite horizon discrete time economy. The set of dates is $\{0, 1, \ldots\}$ and there is no uncertainty at $t = 0$. However, given a history of realizations of the states of nature for the first $t-1$ dates, with $t \geq 1$, $\overline{s}_t = (s_0, \ldots, s_{t-1})$, there is a finite set $S(\overline{s}_t)$ of states of nature that may occur at date t. A vector $\xi = (t, \overline{s}_t, s)$, where $t \geq 1$ and $s \in S(\overline{s}_t)$, is called a *node* of the economy. The only node at $t = 0$ is denoted by ξ_0. Let D be the *event-tree*, i.e., the set of all nodes.

Given $\xi = (t, \overline{s}_t, s)$ and $\mu = (t', \overline{s}_{t'}, s')$, we say that μ is a *successor* of ξ, and we write $\mu \geq \xi$, if $t' \geq t$ and $\overline{s}_{t'} = (\overline{s}_t, s, \ldots)$. We write $\mu > \xi$ to say that $\mu \geq \xi$ but $\mu \neq \xi$ and we denote by $t(\xi)$ the date associated with a node ξ. Let $\xi^+ = \{\mu \in D : (\mu \geq \xi) \wedge (t(\mu) = t(\xi) + 1)\}$ be the set of immediate successors of ξ. The (unique) predecessor of ξ is denoted by ξ^- and $D(\xi) := \{\mu \in D : \mu \geq \xi\}$ is the sub-tree with root ξ.

At each node, a finite set of perishable commodities is available for trade, L. Let $p = (p(\xi); \xi \in D)$, where $p(\xi) := (p(\xi, l); l \in L)$ denotes the commodity price at $\xi \in D$. We assume that there is only one asset, *money*, that can be traded at any node along the event-tree. Although this security does not deliver any payment, it can be used to make intertemporal transfers. Let $q = (q(\xi); \xi \in D)$ be the plan of state-dependent monetary prices. We assume that money is in positive net supply that does not disappear from the economy neither depreciates.

A finite number of agents, $h \in H$, can trade money and buy commodities along the event-tree. Agent h is characterized by his physical and financial endowments, $(w^h(\xi), e^h(\xi)) \in \mathbb{R}^L_{++} \times \mathbb{R}_+$, at each $\xi \in D$, and by his preferences over consumption, which are represented by an utility function $U^h : \mathbb{R}^{D\times L}_+ \to \mathbb{R}_+ \cup \{+\infty\}$. For any $\xi \in D$, let $W_\xi = \sum_{h\in H} w^h(\xi)$ be the aggregated physical endowment at node ξ.

The consumption allocation of agent h at $\xi \in D$ is denoted by $x^h(\xi) := (x^h(\xi, l); l \in L)$. Analogously, the number $z^h(\xi)$ denotes the quantity of money that h negotiates at ξ. Thus, if $z^h(\xi) > 0$, he buys the asset, otherwise, he short sales money making future promises.

Given prices (p, q), let $B^h(p, q)$ be the choice set of agent $h \in H$, that is, the set of plans $(x, z) := ((x(\xi), z(\xi)); \xi \in D) \in \mathbb{R}^{D\times L}_+ \times \mathbb{R}^D$, such that, at each $\xi \in D$, the following budget and debt constraints hold,

$$
\begin{aligned}
&g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) := \\
&\qquad p(\xi)\left(x^h(\xi) - w^h(\xi)\right) + q(\xi)\left(z^h(\xi) - e^h(\xi) - z^h(\xi^-)\right) \leq 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q(\xi)z^h(\xi) + p(\xi)M \geq 0,
\end{aligned}
$$

where $y^h(\xi) = (x^h(\xi), z^h(\xi))$, $z^h(\xi_0^-) = 0$ and $M \in \mathbb{R}^L_+$. Note that short sales of money are bounded by the exogenous *debt constraints* above in order to avoid Ponzi schemes. Agent's h individual problem is to choose a plan $y^h = (x^h, z^h)$ in $B^h(p, q)$ in order to maximize his utility functions U^h.

Definition 1. *An equilibrium for our economy is given by a vector of prices (p, q) jointly with individual allocations $\left((x^h, z^h); h \in H\right)$, such that,*

(a) For each $h \in H$, the plan $(x^h, z^h) \in B^h(p, q)$ is optimal, at prices (p, q),

(b) Physical and asset markets clear,

$$
\sum_{h\in H}\left(x^h(\xi); z^h(\xi)\right) = \left(W_\xi, \sum_{h\in H}\left(e^h(\xi) + z^h(\xi^-)\right)\right).
$$

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2.3 Characterizing Monetary Equilibria

In our economy, a *pure spot market equilibrium*, i.e., an equilibrium with zero monetary price, always exists provided that preferences satisfy the first part of the following hypothesis. However, our objective is to determine conditions that characterize the existence of equilibria with positive price of money, called *monetary equilibria*. For this reason, we also assume that agents are *uniformly impatience*.

Assumption A.
A1. Preferences. *Let* $U^h(x) := \sum_{\xi\in D} u^h(\xi, x(\xi))$, *where for any* $\xi \in D$, $u^h(\xi,\cdot) : \mathbb{R}^L_+ \to \mathbb{R}_+$ *is a continuous, concave and strictly increasing function. Also,* $\sum_{\xi\in D} u^h(\xi, W_\xi)$ *is finite.*

A2. Uniform Impatience Assumption. *There are* $\pi \in [0,1)$ *and* $(v(\mu);\mu \in D) \in \mathbb{R}^{D\times L}_+$ *such that, given a consumption plan* $(x(\mu);\mu \in D)$, *with* $0 \le x(\mu) \le W_\mu$, *for any* $h \in H$, *we have*

$$u^h(\xi, x(\xi) + v(\xi)) + \sum_{\mu>\xi} u^h(\mu,\, \pi'\, x(\mu)) > \sum_{\mu\ge\xi} u^h(\mu, x(\mu)), \qquad \forall \xi \in D,\ \forall \pi' \ge \pi.$$

Moreover, there is $\delta > 0$ *such that,* $w^h(\xi) \ge \delta v(\xi)$, $\forall \xi \in D$.

The requirements of impatience above depend on both preferences and physical endowments. As particular cases we obtain the assumptions imposed by Hernandez & Santos (1996) and Magill & Quinzii (1996). Indeed, in Hernandez & Santos (1996), for any $\mu \in D$, $v(\mu) = W_\mu$. Also, since in Magill & Quinzii (1996) initial endowments are uniformly bounded away from zero by an interior bundle $\underline{w} \in \mathbb{R}^L_+$, they suppose that $v(\mu) = (1, 0, \ldots, 0)$, $\forall \mu \in D$.

Under Assumption A1, Propositions A1 and B1 in the Appendices assure that, given an equilibrium $\left[(p,q);((x^h,z^h);h\in H)\right]$, there are, for each $h \in H$, Kuhn-Tucker multipliers $(\gamma^h(\xi);\xi \in D)$, such that,

$$q(\xi) = F(\xi, q, \gamma^h) + \lim_{T\to+\infty} \sum_{\{\mu\ge\xi: t(\mu)=T\}} \frac{\gamma^h(\mu)}{\gamma^h(\xi)} q(\mu),$$

where $F(\xi, q, \gamma^h)$ is the *fundamental value* of money, and the second term in the right hand side is the monetary speculative component, also called *bubble*. We say that *debt constraints induce frictions over agent* h *in* $\tilde{D} \subset D$ if the

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plan of shadow prices $(\eta^h(\mu);\ \mu \in \tilde{D})$ that is defined implicitly, at each $\mu \in \tilde{D}$, by the conditions:

$$\begin{aligned} 0 &= \eta^h(\mu)\left(q(\mu)z^h(\mu) + p(\mu)M\right), \\ \gamma^h(\mu)q(\mu) &= \sum_{\nu\in\mu^+} \gamma^h(\nu)q(\nu) + \eta^h(\mu)q(\mu), \end{aligned}$$

is different from zero.

Just before our main result, we will show, in the following example, that a monetary equilibrium can arise in economies without uncertainty. Essentially, it is an extension of the monetary model of Bewley (1980) (for references over this example, see Mattalia (2003)) modifying the no short-sales constraint. However, we show that, in this monetary equilibrium, there is no bubble component, only a fundamental value component which is a consequence of the binding no short-sale constraints.

Example 1. Consider an infinite-horizon economy without uncertainty. There is only one asset, fiat money, and a single perishable good at each date. There are two agents $i \in \{1,2\}$ with identical utilities of the form

$$U^i(x_0^i, x_1^i, \ldots) = \sum_{t=0}^{\infty} \beta^t u(x_t^i),$$

where $\beta \in (0,1)$ denotes the intertemporal discounted factor and the function $u : \mathbb{R} \to \mathbb{R}$ is strictly increasing, continuous and strictly concave in $\mathbb{R}_+$.

Each agent i will receive an endowment of money, $e^i \geq 0$, only at the initial date $t = 0$, where $e_0^1 = 0$ and $e_0^2 = 1$. Moreover, the physical endowments of agents at date t are strictly positive and given by $(w_t^1, w_t^2) := (\overline{w}, \underline{w})$, when t is even; and by $(w_t^1, w_t^2) := (\underline{w}, \overline{w})$, when t is odd. Note that $\underline{w} < \overline{w}$.

Finally, the amount of debt each agent can take is bounded, at each date t, by a constraint $M^i(t, p(t)) := p(t)M$, where $p(t)$ denotes the price of the single commodity at t and $M = 0$ is the borrowing limit[3]. The price of money at date t will be denoted by $q(t)$.

We affirm that the prices $q(t) = q^* > 0$ (and $p(t) = 1$) constitute an equilibrium monetary price. Also, the consumption and portfolio allocations

[3]It should be noted that this example could be extended to a fixed $M > 0$. The objective here, however, is to keep the example as simple as possible.

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$$x_t^i = \begin{cases} \overline{w} - q^* & \text{if } i = 1 \text{ and t is even or } i = 2 \text{ and t is odd;} \\ \underline{w} + q^* & i = 1 \text{ and t is odd or } i = 2 \text{ and t is even.} \end{cases}$$

$$z_t^i = \begin{cases} 0 & \text{if } i = 1 \text{ and t is even or } i = 2 \text{ and t is odd;} \\ 1 & i = 1 \text{ and t is odd or } i = 2 \text{ and t is even.} \end{cases}$$

where x_t^i is the consumption choice and z_t^i is the portfolio choice. They are budget and market feasible.

Also, they are optimal choices for the agents as the following Euler conditions are satisfied (see Definition A1 and Proposition A2 in the Appendix A),

$$(\gamma_t^i, \gamma_t^i q^*) = (\beta^t u'(x_t^i), \gamma_{t+1}^i q^* + \lambda_t^i q^*),$$

where γ_t^i is the candidate for Kuhn-Tucker multiplier and λ_t^i is the candidate for shadow price associated to the debt constraint.

Of course, as $x_t^i > 0$, we know that $\gamma_t^i = \beta^t u'(x_t^i)$. Then, it can be shown that, under Kuhn-Tucker multipliers, transversality condition is satisfied and that there is no bubble component. Under Kuhn-Tucker multipliers, the discounted value of future wealth is also finite.

Note that at each t, debt constraint is binding for one of the agents: when t is even, agent 1 binds his constraint. When t is odd, agent 2 portfolio choice is $z_t^2 = 0$. As a consequence, there is a monetary equilibrium that is actually a fundamental value. Money has a positive value as a consequence of the positive shadow values associated to the binding debt constraints.

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Theorem. *Under Assumption A, for any equilibrium* $\big[(p,q);((x^h,z^h);h\in H)\big]$ *we have that,*

(1) If $q(\xi) > 0$ *then debt constraints induce frictions over each agent in* $D(\xi)$.

(2) If $M \neq 0$ *and some* $h \in H$ *has a binding debt constraint at a node* $\mu \in D(\xi)$*, then* $q(\xi) > 0$.

(3) If for each $\xi \in D$, $u^h(\xi,\cdot)$ *is differentiable in* $\mathbb{R}^L_{++}$ *and* $\lim_{\|x\|_{min}\to 0^+} \nabla u^h(\xi,x) = +\infty$, *then any monetary equilibrium is Pareto inefficient.*

Observation. Item (1) is related to Theorem 3.3 in Santos & Woodford (1997), that establishes that, under uniform impatience, assets in positive net supply are free of price bubbles for deflators that yield finite present value of wealth. However, in the frictionless framework used by these authors, absence of bubbles necessarily leads to a zero price of money. The converse, item (2) and item (3) are new in the literature.

Moreover, it follows from items (1) and (2) that binding debt constraints always induce frictions, i.e., positive shadow prices. Also, if an agent becomes borrower at some node in $D(\xi)$, then *all individuals* are borrowers at some node of $D(\xi)$. In other words, in a monetary equilibrium, all agents take a monetary loan (at some node).

PROOF OF THE THEOREM. (1) By definition, if for some $h \in H$, $(\eta^h(\mu); \mu \geq \xi) = 0$ then $F(\xi, q, \gamma^h) = 0$. Therefore, as in Santos & Woodford (1997), a monetary equilibrium is a *pure bubble*. However, uniform impatience implies that bubbles are ruled out in equilibrium.

Indeed, at each $\xi \in D$ there exists an agent $h = h(\xi)$ with $q(\xi)z^h(\xi) \geq 0$. Thus, by the impatience property, $0 \leq (1-\pi)q(\xi)z^h(\xi) \leq p(\xi)v(\xi)$. Moreover, financial market feasibility allows us to find a lower bound for individual debt. Therefore, for each $h \in H$, the plan $\left(\frac{q(\xi)z^h(\xi)}{p(\xi)v(\xi)}\right)_{\xi \in D}$ is uniformly bounded. Furthermore, as money is in positive net supply, it follows that $\left(\frac{q(\xi)}{p(\xi)v(\xi)}\right)_{\xi \in D}$ is uniformly bounded too. As by Lemma A1 we know that, for any $h \in H$, $\sum_{\xi \in D} \gamma^h(\xi)p(\xi)w^h(\xi) < +\infty$, it follows from Assumption A2 that bubbles do not arise in equilibrium.

Therefore, we conclude that, if $q(\xi) > 0$ then $(\eta^h(\mu); \mu \geq \xi) \neq 0$, for all $h \in H$.

(2) Suppose that, for some $h \in H$, there exists $\mu \geq \xi$ such that $q(\mu)z^h(\mu) = -p(\mu)M$. Since monotonicity of preferences implies that $p(\xi) \gg 0$, if $M \neq 0$ then $q(\mu) > 0$. Also, Assumption A1 assures that Kuhn-Tucker multipliers, $(\gamma^h(\eta); \eta \in D)$, are strictly positive. Therefore, the equations that define shadow prices imply that $q(\xi) > 0$.

(3) Suppose that there exists an efficient monetary equilibrium, in the sense that individuals' marginal rates of substitution coincide. As $\lim_{\|x\|_{min} \to 0^+} \nabla u^h(\xi, x) = +\infty, \forall (h, \xi) \in H \times D$, all agents have interior consumption along the event-tree. Positive net supply of money implies that there exists, at each $\xi \in D$, at least one lender. Therefore, by the efficiency property, it follows that *all* individuals have zero shadow prices. A contradiction with item (1) above. $\square$

Some remarks,

(1) It follows from the proof of the Theorem that under uniform impatience

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monetary debt is uniformly bounded—in real terms—along the even-tree. Thus, it is easy to find a vector $M^* \in \mathbb{R}^L_+$ such that, in any equilibrium, and for each node ξ, the debt constraint $q(\xi)z^h(\xi) \geq -p(\xi)M^*$ is non-binding. Therefore, when $M > M^*$ monetary equilibria disappear. That is, contrary to what may be expected, frictions induced by debt constraints improve welfare.

(2) Given a monetary equilibrium, there always exists a non-arbitrage deflator, *incompatible with physical Euler conditions* (see Definition A1), for which the price of money is a pure bubble. Indeed, define $\nu := (\nu(\xi) : \xi \in D)$ by $\nu(\xi_0) = 1$, and

$$\begin{aligned} \nu(\xi) &= 1, & \forall \xi > \xi_0 : q(\xi) = 0, \\ \frac{\nu(\xi)}{\nu(\xi^-)} &= \frac{\gamma^h(\xi)}{\gamma^h(\xi^-) - \eta^h(\xi^-)}, & \forall \xi > \xi_0 : q(\xi) > 0. \end{aligned}$$

Euler conditions on $(\gamma^h(\xi);\, \xi \in D)$ imply that, for each $\xi \in D$, $\nu(\xi)q(\xi) = \sum_{\mu \in \xi^+} \nu(\mu)q(\mu)$. Therefore, using the plan of deflators ν, *financial Euler conditions* hold and the positive price of money is a bubble. Since under uniform impatience assumption the monetary debt is uniformly bounded along the event-tree, under these deflators the discounted value of future individual endowments has to be infinite.

We remark that the plan of state prices ν is compatible with the frictionless theory of bubbles developed by Santos & Woodford (1997) and, in that frictionless context, we recover a property that was previously found by them: a monetary bubble is possible only for deflators under which we have an infinite discounted value of future wealth.

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2.4
About Uniform Impatience

To highlight the role that uniform impatience has in our Theorem, we adapt Example 1 in Araujo, Páscoa & Torres-Martínez (2007) in order to prove that without Assumption A2 money may have a pure bubble for Kuhn-Tucker multipliers. Moreover, bubbles on the price of money will be compatible with a finite discounted value of future wealth. Essentially because individuals will believe that, as time goes on, the probability that the economy may fall in a path in which endowments increase without an upper bound converges to zero fast enough.

Example 2. Assume that each $\xi \in D$ has two successors: $\xi^+ = \{\xi^u, \xi^d\}$. There are two agents $H = \{1, 2\}$ and only one commodity. Each $h \in H$

has physical endowments $(w^h_\xi)_{\xi\in D}$, receives financial endowments $e^h \geq 0$ only at the first node, and has preferences represented by the utility function $U^h(x) = \sum_{\xi\in D} \beta^{t(\xi)}\rho^h(\xi)\, x_\xi$, where $\beta \in (0,1)$ and the plan $(\rho^h(\xi))_{\xi\in D} \in (0,1)^D$ satisfies $\rho(\xi_0) = 1$, $\rho^h(\xi) = \rho^h(\xi^d) + \rho^h(\xi^u)$ and

$$\rho^1(\xi^u) = \frac{1}{2^{t(\xi)+1}}\rho^1(\xi), \qquad\qquad \rho^2(\xi^u) = \left(1 - \tfrac{1}{2^{t(\xi)+1}}\right)\, \rho^2(\xi).$$

Suppose that agent $h = 1$ is the only one endowed with the asset, i.e., $(e^1, e^2) = (1, 0)$ and that, for each $\xi \in D$,

$$w^1_\xi = \begin{cases} 1 + \beta^{-t(\xi)} & \text{if } \xi \in D^{du}, \\ 1 & \text{otherwise}\,; \end{cases} \qquad\qquad w^2_\xi = \begin{cases} 1 + \beta^{-t(\xi)} & \text{if } \xi \in \{\xi^d_0\} \cup D^{ud}, \\ 1 & \text{otherwise}\,; \end{cases}$$

where D^{du} is the set of nodes attained after going down followed by up, that is, $D^{du} = \{\eta \in D :\, \exists\xi,\;\; \eta = (\xi^d)^u\;\; \}$ and D^{ud} denotes the set of nodes reached by going up and then down, that is, $D^{ud} = \{\eta \in D :\, \exists\xi,\;\; \eta = (\xi^u)^d\;\; \}$.

Agents will use positive endowment shocks in low probability states to buy money and sell it later in states with higher probabilities. Let prices be $(p_\xi, q_\xi)_{\xi\in D} = (\,\beta^{t(\xi)},\, 1)_{\xi\in D}$ and suppose that consumption of agent h is given by $x^h_\xi = w^{h'}_\xi$, where $h \neq h'$. It follows from budget constraints that, at each ξ, the portfolio of agent h must satisfy $z^h_\xi = \beta^{t(\xi)}(w^h_\xi - w^{h'}_\xi) + z^h_{\xi^-}$, where $z^h_{\xi_0^-} := e^h$ and $h \neq h'$.

Thus, the consumption allocations above jointly with the portfolios $(z^1_{\xi_0},\, z^1_{\xi^u},\, z^1_{\xi^d}) = (1,\, 1,\, 0)$ and $(z^2_\xi)_{\xi\in D} = (1 - z^1_\xi)_{\xi\in D}$ are budget and market feasible. Finally, given $(h,\xi) \in H \times D$, let $\gamma^h_\xi = \rho^h(\xi)$ be the candidate for Kuhn-Tucker multiplier of agent h at node ξ. It follows that conditions below hold and they assure individual optimality (see Proposition A2 in the Appendix A),

$$\begin{aligned} (\gamma^h_\xi p_\xi, \gamma^h_\xi q_\xi) \quad &= \quad (\beta^{t(\xi)}\rho^h(\xi), \gamma^h_{\xi^u} q_{\xi^u} + \gamma^h_{\xi^d} q_{\xi^d}), \\ \sum_{\{\eta\in D:\, t(\eta)=T\}} \gamma^h_\eta p_\eta M \longrightarrow \quad &0, \qquad \text{as } T \to +\infty, \\ \sum_{\{\eta\in D:\, t(\eta)=T\}} \gamma^h_\eta q_\eta z^h_\eta \longrightarrow \quad &0, \qquad \text{as } T \to +\infty. \end{aligned}$$

Note that, by construction and independently of $M \geq 0$, the plan of shadow prices associated to debt constraints is zero. Therefore, for any M, money has a zero fundamental value and a bubble under Kuhn-Tucker multipliers. Also, the diversity of individuals beliefs about the uncertainty (probabilities $\rho^h(\xi)$) implies that both agents perceive a finite present value of

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aggregate wealth.[4] Finally, Assumption A2 is not satisfied, because aggregated physical endowments were unbounded along the event-tree.[5] □

Appendix A: Duality Theory of Individual Optimality

Under Assumption A1, we will use duality theory to determine necessary conditions for individual optimality. To attempt this objective, we restrict our attention, without loss of generality, to prices $(p,q) \in \mathbb{P} := \{(p,q) \in \mathbb{R}_+^{L\times D} \times \mathbb{R}_+^D : (p(\xi),q(\xi)) \in \Delta^{\#L+1}, \ \forall \xi \in D\}$, where, for each $m > 0$, the *simplex* $\Delta^m := \{z = (z_1,\ldots,z_m) \in \mathbb{R}_+^m : \sum_{k=1}^m z_k = 1\}$. Also, remember that the super-gradient of a concave function $f : X \subset \mathbb{R}^L \to \mathbb{R} \cup \{-\infty\}$ at point $x \in X$ is defined as the set of vectors $p \in \mathbb{R}^L$ such that, for all $x' \in X$, $f(\xi,x') - f(\xi,x) \leq p(x'-x)$.

For convenience of notations, let the set of nodes with date T in $D(\xi)$ be denoted by $D_T(\xi)$. Finally, let $D^T(\xi) = \bigcup_{k=t(\xi)}^T D_k(\xi)$ be the set of successors of ξ with date less than or equal to T. When $\xi = \xi_0$ notations above will be shorten to D_T and D^T.

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[4]Using agent' h Kuhn-Tucker multipliers as deflators, the present value of aggregated wealth at $\xi \in D$, denoted by PV_ξ^h, satisfies,

$$
\begin{aligned}
PV_\xi^h &= \sum_{\mu \geq \xi} \frac{\gamma_\mu^h}{\gamma_\xi^h} p_\mu W_\mu = \frac{2}{\rho^h(\xi)} \sum_{\mu\geq\xi} \rho^h(\mu)\beta^{t(\mu)} + \frac{1}{\rho^h(\xi)} \sum_{\{\mu\geq\xi : \mu \in D^{ud}\cup D^{du}\cup\{\xi_0^d\}\}} \rho^h(\mu) \\
&= 2\frac{\beta^{t(\xi)}}{1-\beta} + \sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du}\cup\{\xi_0^d\},\, t(\mu)\leq t(\xi)+1\}} \frac{\rho^h(\mu)}{\rho^h(\xi)} \\
&+ \sum_{s=t(\xi)+1}^{+\infty} \left[\frac{1}{2^{s+1}}\left(1-\frac{1}{2^s}\right) + \left(1-\frac{1}{2^{s+1}}\right)\frac{1}{2^s}\right] \\
&= 2\frac{\beta^{t(\xi)}}{1-\beta} + \frac{3}{2}\frac{1}{2^{t(\xi)}} - \frac{1}{3}\frac{1}{4^{t(\xi)}} + \frac{1}{\rho^h(\xi)} \sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du},\, t(\mu)\leq t(\xi)+1\}} \rho^h(\mu) < +\infty.
\end{aligned}
$$

[5]If Assumption A2 holds, there are $(\delta,\pi) \in \mathbb{R}_{++} \times (0,1)$ such that, for any $\xi \in D^{uu} := \{\mu \in D : \exists \eta \in D; \mu = (\eta^u)^u\}$,

$$\frac{1}{\delta} = \frac{w_\xi^h}{\delta} > \frac{1-\pi}{\beta^{t(\xi)}\rho^h(\xi)} \sum_{\mu>\xi} \rho^h(\mu)\beta^{t(\mu)} W_\mu, \quad \forall h \in H.$$

Thus, for all $(\xi,h) \in D^{uu} \times H$, $\beta^{t(\xi)}\left(\frac{1}{\delta(1-\pi)} + W_\xi\right) > PV_\xi^h$. On the other hand, given $\xi \in D^{uu}$,

$$PV_\xi^1 \geq \frac{1}{\rho^1(\xi)} \sum_{\{\mu\geq\xi:\mu\in D^{ud}\cup D^{du},\, t(\mu)\leq t(\xi)+1\}} \rho^1(\mu) = 1 - \frac{1}{2^{t(\xi)+1}}.$$

Therefore, as for any $T \in \mathbb{N}$ there exists $\xi \in D^{uu}$ with $t(\xi) = T$, we conclude that, $\beta^T\left(\frac{1}{\delta(1-\pi)} + 2\right) > 0.5$, for all $T > 0$. A contradiction.

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Definition A1. *Given* $(p,q) \in \mathbb{P}$ *and* $y^h = (x^h, z^h) \in B^h(p,q)$, *we say that* $(\gamma^h(\xi); \xi \in D) \in \mathbb{R}_+^D$ *constitutes a family of Kuhn-Tucker multipliers (associated to* y^h*) if there exist, for each* $\xi \in D$, *super-gradients* $u'(\xi) \in \partial u^h(\xi, x^h(\xi))$ *such that,*

(a) For every $\xi \in D$, $\gamma^h(\xi)\, g_\xi^h(y^h(\xi), y^h(\xi^-); p, q) = 0$.

(b) The following Euler conditions hold,

$$\begin{aligned} \gamma^h(\xi)p(\xi) &\geq u'(\xi), \\ \gamma^h(\xi)p(\xi)x^h(\xi) &= u'(\xi)\, x^h(\xi), \\ \gamma^h(\xi)q(\xi) &\geq \sum_{\mu \in \xi^+} \gamma^h(\mu)q(\mu), \end{aligned}$$

where the last inequality is strict only if the associated debt constraint is binding at ξ.

(c) The following transversality condition holds:

$$\limsup_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi)q(\xi)z^h(\xi) \leq 0.$$

Lemma A1. (Finite discounted value of individual endowments)

Fix a plan $(p,q) \in \mathbb{P}$ *and* $y^h = (x^h, z^h) \in B^h(p,q)$ *such that* $U^h(x^h) < +\infty$. *If Assumption A1 holds then for any family of Kuhn-Tucker multipliers associated to* y^h, $(\gamma^h(\xi); \xi \in D)$, *we have* $\sum_{\xi \in D} \gamma^h(\xi) \left(p(\xi)\, w^h(\xi) + q(\xi)e^h(\xi)\right) < +\infty$.

Proof. Let $\mathcal{L}_\xi^h : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the function defined by $\mathcal{L}_\xi^h(y(\xi), y(\xi^-)) = v^h(\xi, y(\xi)) - \gamma^h(\xi)\, g_\xi^h(y(\xi), y(\xi^-); p, q)$, where $y(\xi) = (x(\xi), z(\xi))$ and $v^h(\xi, \cdot) : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is given by

$$v^h(\xi, y(\xi)) = \begin{cases} u^h(\xi, x(\xi)) & \text{if } x(\xi) \geq 0; \\ -\infty & \text{otherwise.} \end{cases}$$

It follows from Assumption A1 and Euler conditions that, for each $T \geq 0$,

$$\sum_{\xi \in D^T} \mathcal{L}_\xi^h(0,0) - \sum_{\xi \in D^T} \mathcal{L}_\xi^h(y^h(\xi), y^h(\xi^-)) \leq -\sum_{\xi \in D_T} \gamma^h(\xi)q(\xi)(0 - z^h(\xi)).$$

Therefore, as for each $\xi \in D$, $\gamma^h(\xi)\, g_\xi^h(y^h(\xi), y^h(\xi^-); p, q) = 0$, we have that, for any $S \in \mathbb{N}$,

$$0 \leq \sum_{\xi \in D^S} \gamma^h(\xi) \left(p(\xi)w^h(\xi) + q(\xi)e^h(\xi)\right)$$

$$\begin{aligned}
&\leq \limsup_{T \to +\infty} \sum_{\xi \in D^T} \gamma^h(\xi) \left(p(\xi)w^h(\xi) + q(\xi)e^h(\xi)\right) \\
&\leq U^h(x^h) + \limsup_{T} \sum_{\xi \in D_T} \gamma^h(\xi)q(\xi)z^h(\xi) \\
&\leq U^h(x^h) < +\infty,
\end{aligned}$$

which concludes the proof. □

Proposition A1. (Necessary conditions for individual optimality) *Fix a plan $(p,q) \in \mathbb{P}$ and $y^h = (x^h, z^h) \in B^h(p,q)$ such that $U^h(x^h) < +\infty$. If Assumption A1 holds and y^h is an optimal allocation for agent $h \in H$ at prices (p,q), then there exists a family of Kuhn-Tucker multipliers associated to y^h.*

Proof. Suppose that $(y^h(\xi))_{\xi \in D}$ is optimal for agent $h \in H$ at prices (p,q). For each $T \in \mathbb{N}$, consider the truncated optimization problem,

$$(P^{h,T}) \qquad \begin{array}{ll} \max & \sum_{\xi \in D^T} u^h(\xi, x(\xi)) \\ \text{s.t.} & \begin{cases} g_\xi^h\left(y(\xi), y(\xi^-); p, q\right) & \leq \; 0, \forall \xi \in D^T, \\ q(\xi)z(\xi) & \geq \; -p(\xi)M, \forall \xi \in D^T \setminus D_T, \\ (x(\xi), z(\eta)) & \geq \; 0, \; \forall (\xi, \eta) \in D^T \times D_T. \end{cases} \end{array}$$

It follows that, under Assumption A1 each truncated problem $P^{h,T}$ has a solution $(y^{h,T}(\xi))_{\xi \in D^T}$.[6]

[6] In fact, as $(y^h(\xi))_{\xi \in D}$ is optimal and $U^h(x^h) < +\infty$, it follows that there exists a solution for $P^{h,T}$ if and only if there exists a solution for the problem,

$$(\tilde{P}^{h,T}) \qquad \begin{array}{ll} \max & \sum_{\xi \in D^T} u^h(\xi, x(\xi)) \\ \text{s.t.} & \begin{cases} g_\xi^h\left(y(\xi), y(\xi^-); p, q\right) & \leq \; 0, \forall \xi \in D^T, \text{ where } y(\xi) = (x(\xi), z(\xi)), \\ z(\xi) & \geq \; -\frac{p(\xi)M}{q(\xi)}, \; \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0 \\ z(\xi) & = \; 0, \text{if } \left[\xi \in D^{T-1} \text{ and } q(\xi) = 0\right] \text{or } \xi \in D_T, \\ x(\xi) & \geq \; 0, \forall \xi \in D^T. \end{cases} \end{array}$$

Indeed, it follows from the existence of an optimal plan which gives finite utility that if $q(\xi) = 0$ for some $\xi \in D$, then $q(\mu) = 0$ for each successor $\mu > \xi$. Now, budget feasibility assures that,

$$z(\xi) \leq \frac{p(\xi)w^h(\xi)}{q(\xi)} + z(\xi^-), \; \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0.$$

As $z(\xi_0^-) = 0$, the set of feasible financial positions is bounded in the problem $(\tilde{P}^{h,T})$. Thus, budget feasible consumption allocations are also bounded and, therefore, the set of admissible strategies is compact. As the objective function is continuous, there is a solution for $(\tilde{P}^{h,T})$. Moreover, the optimality of $(y^h(\xi))_{\xi \in D}$ in the original problem implies that

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Given a multiplier $\gamma \in \mathbb{R}$, let $\mathcal{L}^h_\xi(\cdot, \gamma; p, q) : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the Lagrangian at node ξ, i.e.,

$$\mathcal{L}^h_\xi(y(\xi), y(\xi^-), \gamma;\, p, q) = v^h(\xi, y(\xi)) - \gamma\, g^h_\xi(y(\xi), y(\xi^-); p, q).$$

It follows from Rockafellar (1997, Theorem 28.3) that there exist non-negative multipliers $(\gamma^{h,T}(\xi))_{\xi \in D^T}$ such that the following saddle point property

$$\sum_{\xi \in D^T} \mathcal{L}^h_\xi(y(\xi), y(\xi^-), \gamma^{h,T}(\xi);\, p, q) \le \sum_{\xi \in D^T} \mathcal{L}^h_\xi(y^{h,T}(\xi), y^{h,T}(\xi^-), \gamma^{h,T}(\xi);\, p, q),$$

is satisfied, for each plan $(y(\xi))_{\xi \in D^T} = (x(\xi), z(\xi))_{\xi \in D^T}$ for which

$$\begin{aligned} (x(\xi), z(\eta)) &\ge 0, && \forall (\xi, \eta) \in D^T \times D_T, \\ q(\xi) z(\xi) &\ge -p(\xi) M, && \forall \xi \in D^T \setminus D_T. \end{aligned}$$

Moreover, at each $\xi \in D^T$, multipliers satisfy $\gamma^{h,T}(\xi)\, g^h_\xi(y^{h,T}(\xi), y^{h,T}(\xi^-); p, q) = 0$.

Analogous arguments to those made in Claims A1-A3 in Araujo, Páscoa & Torres-Martínez (2007) imply that,

Claim. *Under Assumption A1, the following conditions hold:*
(i) For each $t < T$,

$$0 \le \sum_{\xi \in D^t} \gamma^{h,T}(\xi) \left(p(\xi) w^h(\xi) + q(\xi) e^h(\xi) \right) \le U^h(x^h).$$

(ii) For each $0 < t < T$,

$$\sum_{\xi \in D_t} \gamma^{h,T}(\xi) q(\xi) z^h(\xi^-) \le \sum_{\xi \in D \setminus D^{t-1}} u^h(\xi, x^h(\xi)).$$

(iii) For each $\xi \in D^{T-1}$ and for any $y(\xi) = (x(\xi), z(\xi))$, with $x(\xi) \ge 0$ and $q(\xi) z(\xi) \ge -p(\xi) M$,

$U^h(x^h)$ is greater than or equal to $\sum_{\xi \in D^T} u^h(\xi, x^{h,T}(\xi))$. In fact, the plan $(\tilde{y}_\xi)_{\xi \in D}$ defined by $\tilde{y}_\xi = y^{h,T}_\xi$, for each $\xi \in D^T$, and by $\tilde{y}_\xi = 0$ otherwise, is budget feasible in the original economy and, therefore, the allocation $(y^{h,T}(\xi))_{\xi \in D^T}$ cannot improve the utility level of agent h.

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$$u^h(\xi, x(\xi)) - u^h(\xi, x^h(\xi)) \leq$$

$$\left(\gamma^{h,T}(\xi)p(\xi); \gamma^{h,T}(\xi)q(\xi) - \sum_{\mu\in\xi^+}\gamma^{h,T}(\mu)q(\mu)\right) \cdot (y(\xi) - y^h(\xi))$$

$$+ \sum_{\eta\in D\setminus D^T} u^h(\eta, x^h(\eta)).$$

Now, at each $\xi \in D$, $\underline{w}^h(\xi) := \min_{l\in L} w^h(\xi, l) > 0$. Also, as a consequence of monotonicity of $u^h(\xi)$, $||p(\xi)||_\Sigma > 0$. Thus, item (i) above guarantees that, for each $\xi \in D$,

$$0 \leq \gamma^{h,T}(\xi) \leq \frac{U^h(x^h)}{\underline{w}^h(\xi)\,||p(\xi)||_\Sigma}, \qquad \forall T > t(\xi).$$

Therefore, the sequence $(\gamma^{h,T}(\xi))_{T\geq t(\xi)}$ is bounded, node by node. As the event-tree is countable, there is a common subsequence $(T_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and non-negative multipliers $(\gamma^h(\xi))_{\xi\in D}$ such that, for each $\xi \in D$, $\gamma^{h,T_k}(\xi) \to_{k\to+\infty} \gamma^h(\xi)$, and

$$\begin{aligned} \gamma^h(\xi)g^h_\xi(p, q, y^h(\xi), y^h(\xi^-)) &= 0\,; && \text{(2-1)} \\ \limsup_{t\to+\infty} \sum_{\xi\in D_t} \gamma^h(\xi)q(\xi)z^h(\xi^-) &\leq 0\,, && \text{(2-2)} \end{aligned}$$

where equation (2-1) follows from the strictly monotonicity of $u^h(\xi)$, and equation (2-2) is a consequence of item (ii) (taking the limit as T goes to infinity and, afterwards, the limit in t).

Moreover, using item (iii), and taking the limit as T goes to infinity, we obtain that, for each $y(\xi) = (x(\xi), z(\xi))$, with $x(\xi) \geq 0$ and $q(\xi)z(\xi) \geq -p(\xi)M$,

$$\begin{aligned} u^h(\xi, x(\xi)) - u^h(\xi, x^h(\xi)) &\leq \\ (\gamma^h(\xi)p(\xi)\,;\; \gamma^h(\xi)q(\xi) &- \sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)) \cdot (y(\xi) - y^h(\xi)). \end{aligned}$$

Let $\mathcal{F}^h(\xi, p, q) = \{(x, z) \in \mathbb{R}^L \times \mathbb{R} :\, x \geq 0 \;\wedge\; q(\xi)z \geq -p(\xi)M\}$.

It follows that $\left(\gamma^h(\xi)p(\xi)\,;\, \gamma^h(\xi)q(\xi) - \sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right)$ belongs to the super-differential set of the function $v^h(\xi, \cdot) + \delta(\cdot, \mathcal{F}^h(\xi, p, q))$ at point $y^h(\xi)$, where $\delta(y, \mathcal{F}^h(\xi, p, q)) = 0$, when $y \in \mathcal{F}^h(\xi, p, q)$ and $\delta(y, \mathcal{F}^h(\xi, p, q)) = -\infty$, otherwise. Notice that, for each $y \in \mathcal{F}^h(\xi,p,q)$, $\kappa \in \partial\delta(y,\mathcal{F}^h(\xi,p,q)) \Leftrightarrow 0 \leq k(y'-y),\ \ \forall y' \in \mathcal{F}^h(\xi,p,q)$.

Now, by Theorem 23.8 in Rockafellar (1997), for all $y \in \mathcal{F}^h(\xi,p,q)$, if $v'(\xi)$ belongs to $\partial\left[v^h(\xi,y)+\delta(y,\mathcal{F}^h(\xi,p,q))\right]$ then there exists $\tilde{v}'(\xi) \in \partial v^h(\xi,y)$ such that both $v'(\xi) \geq \tilde{v}'(\xi)$ and $(v'(\xi)-\tilde{v}'(\xi))\cdot(x,q(\xi)z+p(\xi)M)=0$, where $y=(x,z)$. Therefore, it follows that there exists, for each $\xi \in D$, a super-gradient $\tilde{v}'(\xi) \in \partial v^h(\xi,y^h(\xi))$ such that,

$$\left(\gamma^h(\xi)p(\xi)\,;\ \gamma^h(\xi)q(\xi)-\sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right)-\tilde{v}'(\xi)\geq 0,$$

$$\left[\left(\gamma^h(\xi)p(\xi)\,;\ \gamma^h(\xi)q(\xi)-\sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)\right)-\tilde{v}'(\xi)\right] \cdot\ (x^h(\xi),q(\xi)z^h(\xi)+p(\xi)M)\ =\ 0.$$

As $\tilde{v}'(\xi) \in \partial v^h(\xi,y^h(\xi))$ if and only if there is $u'(\xi) \in \partial u^h(\xi,x^h(\xi))$ such that $\tilde{v}'(\xi)=(u'(\xi),0)$, it follows from last inequalities that Euler conditions hold.

On the other side, item (i) in claim above guarantees that, $\sum_{\xi\in D}\gamma^h(\xi)(p(\xi)w^h(\xi)+q(\xi)e^h(\xi)) < +\infty$ and, therefore, equations (2-1) and (2-2) assure that, $\limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi)$

$$\begin{aligned}
&\leq\ \limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)\left(p(\xi)w^h(\xi)+q(\xi)e^h(\xi)+q(\xi)z^h(\xi^-)\right)\\
&\leq\ \limsup_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi^-)\ \ \leq\ \ 0,
\end{aligned}$$

which imply that transversality condition holds. □

Note that we could prove, alternatively, the existence of a state price deflator that satisfies the *financial* Euler equation only using, as Santos & Woodford (1997), non-arbitrage conditions. However, to attempt our objectives we need to assure that Kuhn-Tucker deflators exist, in the sense of Definition A1, and also that the discounted value of endowments, using these deflators, is finite.

On the other hand, as under Kuhn-Tucker multipliers the deflated value of individual endowments is finite, our transversality condition is equivalent to the requirement imposed by Magill & Quinzii (1996), provided that either

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short sales were avoided or individual endowments were uniformly bounded away from zero.

COROLLARY.

Fix $(p,q) \in \mathbb{P}$. Under Assumption A1, given $h \in H$ suppose that either $M = 0$ or there exists $\underline{w} \in \mathbb{R}^L_{++}$ such that, at any $\xi \in D$, $w^h(\xi) \geq \underline{w}$. If y^h is an optimal allocation for agent h at prices (p,q), then for any plan of Kuhn-Tucker multipliers associated to y^h, $(\gamma^h(\xi))_{\xi \in D}$, we have,

$$\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) = 0.$$

PROOF. Let $(\gamma^h(\xi))_{\xi \in D}$ be a plan of Kuhn-Tucker multipliers associated to y^h. We know that the transversality condition of Definition A1 holds. On the other hand, it follows directly from the debt constraint that,

$$\sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) \geq -\sum_{\xi \in D_T} \gamma^h(\xi) p(\xi) M \geq -\left(\max_{l \in L} M_l\right) \sum_{\xi \in D_T} \gamma^h(\xi) \|p(\xi)\|_\Sigma.$$

Therefore, when $M = 0$ we obtain the result. Alternatively, assume that for any $\xi \in D$, $w^h(\xi) \geq \underline{w}$. As by Lemma A1 the sum $\sum_{\xi \in D} \gamma^h(\xi) p(\xi) w^h(\xi)$ is well defined and finite we have that

$$\sum_{\xi \in D} \gamma^h(\xi) \|p(\xi)\|_\Sigma < +\infty.$$

Thus, $\liminf_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) \geq 0$ which implies, using the transversality condition of Definition A1, that $\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi) = 0$. □

We end this Appendix with a result that determines sufficient requirements to assure that a plan of consumption and portfolio allocations is individually optimal. Note that the result below will assure that, when either short-sales were avoided, i.e., $M = 0$, or individual endowments were uniformly bounded away from zero, a budget feasible plan is individually optimal *if and only if* there exists a family of Kuhn-Tucker multipliers associated to it.

PROPOSITION A2. (SUFFICIENT CONDITIONS FOR INDIVIDUAL OPTIMALITY)

Fix a plan $(p,q) \in \mathbb{P}$. Under Assumption A1, suppose that given $y^h = (x^h, z^h) \in B^h(p,q)$, there exists a family of Kuhn-Tucker multipliers $(\gamma^h(\xi); \xi \in$

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D) associated to y^h. Then, if

$$\lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h(\xi)p(\xi)M \quad = \quad 0,$$

then y^h is an optimal allocation for agent h at prices (p, q).

Proof. Note that, under the conditions above

$$\lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)z^h(\xi) = 0.$$

On the other hand, it follows from Euler conditions that, for each $T \geq 0$,

$$\sum_{\xi\in D^T} \mathcal{L}^h_\xi(y(\xi), y(\xi^-), \gamma^h(\xi); p, q) - \sum_{\xi\in D^T} \mathcal{L}^h_\xi(y^h(\xi), y^h(\xi^-), \gamma^h_\xi; p, q)$$

$$\leq - \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)(z(\xi) - z^h(\xi)).$$

Moreover, as at each node $\xi \in D$ we have that $\gamma^h(\xi)g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) = 0$, each budget feasible allocation $y = ((x(\xi), z(\xi)); \xi \in D)$ must satisfy

$$\sum_{\xi\in D^T} u^h(\xi, x(\xi)) - \sum_{\xi\in D^T} u^h(\xi, x^h(\xi)) \leq - \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)(z(\xi) - z^h(\xi)).$$

Now, as the sequence $\left(\sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)z^h(\xi)\right)_{T\in\mathbb{N}}$ converges, it is bounded. Thus,

$$\begin{aligned}
\limsup_{T\to+\infty} \left(- \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)(z(\xi) - z^h(\xi))\right) \quad &\leq \quad \limsup_{T\to+\infty} \left(- \sum_{\xi\in D_T} \gamma^h(\xi)q(\xi)z(\xi)\right) \\
&\leq \quad \lim_{T\to+\infty} \sum_{\xi\in D_T} \gamma^h(\xi)p(\xi)M = 0.
\end{aligned}$$

Therefore,

$$U^h(x) = \limsup_{T\to+\infty} \sum_{\xi\in D^T} u^h(\xi, x(\xi)) \leq U^h(x^h),$$

which guarantees that the allocation $(x^h(\xi), z^h(\xi))_{\xi\in D}$ is optimal. $\square$

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Appendix B: On the Fundamental Value of Money

In the frictionless theory developed by Santos & Woodford (1997), that is, where debt constraints are non saturated, two (equivalent) definitions of the fundamental value of money make economic sense. The fundamental value is either (1) equal to the discounted value of future deliveries that an agent will receive for one unit of money that he buys and keeps forever; (2) equal to the discounted value of rental services, that coincides with deliveries, given the absence of any friction associated to debt constraint.

These concepts do not coincide when frictions are allowed. Thus, we adopt the second definition, that internalizes the role that money has: it allows for intertemporal transfers, although its deliveries are zero.

PROPOSITION B1. (NON-EXISTENCE OF NEGATIVE BUBBLES)

Under Assumption A1, given an equilibrium $\left[(p,q);((x^h,z^h);h\in H)\right]$, *at each node* $\xi\in D$, $q(\xi)\geq F(\xi,q,\gamma^h)$, *where* $(\gamma^h(\xi);\xi\in D)$ *denotes the agent's* h *plan of Kuhn-Tucker multipliers and*

$$F(\xi,q,\gamma^h):=\frac{1}{\gamma^h(\xi)}\sum_{\mu\in D(\xi)}\left(\gamma^h(\mu)q(\mu)-\sum_{\nu\in\mu^+}\gamma^h(\nu)q(\nu)\right),$$

is the fundamental value of money at $\xi\in D$.

PROOF. By Proposition A1, there are, for each agent $h\in H$, non-negative shadow prices $(\eta^h(\xi);\xi\in D)$, satisfying for each $\xi\in D$,

$$\begin{aligned} 0 &= \eta^h(\xi)\left(q(\xi)z^h(\xi)+p(\xi)M\right); \\ \gamma^h(\xi)q(\xi) &= \sum_{\mu\in\xi^+}\gamma^h(\mu)q(\mu)+\eta^h(\xi)q(\xi). \end{aligned}$$

Therefore,

$$\gamma^h(\xi)q(\xi)=\sum_{\mu\geq\xi}\eta^h(\mu)q(\mu)+\lim_{T\to+\infty}\sum_{\mu\in D_T(\xi)}\gamma^h(\mu)q(\mu).$$

As multipliers and monetary prices are non-negative, the infinite sum in the right hand side of equation above is well defined, because its partial sums are increasing and bounded by $\gamma^h(\xi)q(\xi)$. This also implies that the limit of the (discounted) asset price exists. □

Note that the rental services that one unit of money gives at $\mu \in D$ are equal to $q(\mu) - \sum_{\nu \in \mu^+} \frac{\gamma^h(\nu)}{\gamma^h(\mu)} q(\mu)$. Thus, the *fundamental value* of money at a node ξ, as was defined in Proposition B1, coincides with the discounted value of (unitary) future rental services.

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