6
Examples

We experiment our constructions on three submanifolds immersed in the plane: a parabola as an example of a smooth submanifold and a V-function and a cusp for non-smooth submanifolds.

6.1
Test functions

To experiment the concept of $\mathcal{D}$-immersions, we need some test functions in order to evaluate our maps.

To do so, we use combinations of the following standard test function:

$$\phi : t \longmapsto \begin{cases} \frac{\exp(t^2/(t^2-1))}{NORM} & if\ t^2 < 1 \quad , \\ 0 & otherwise \quad . \end{cases}$$

From this test function it is possible to generate many others using a very simple trick: averaging several copies of ϕ translated to different points on the

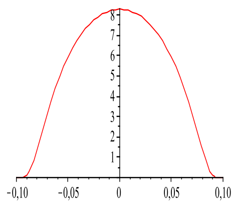
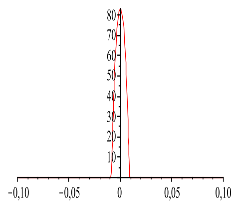
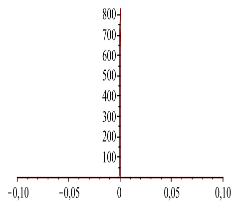
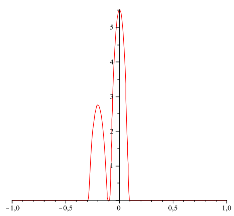
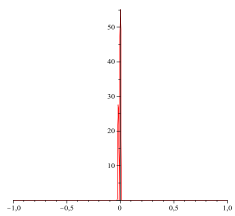
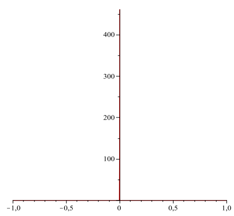
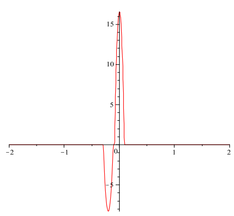
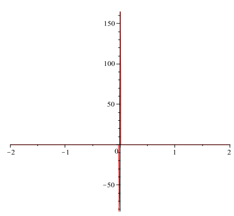
φ	$\epsilon = 0.1$	$\epsilon = 0.01$	$\epsilon = 0.001$
$\phi(t)$			
$\frac{2}{3}\phi(t) + \frac{1}{3}\phi(t+2)$			
$2\phi(t) - \phi(t+2)$			

Table 6.1: Some of the approximations of the identity used in the examples.

real line we get a new test function (see Table 6.1). We can easily then create an approximation of the identity by:

$$\varphi_\epsilon : t \longmapsto \frac{1}{\epsilon^d} \cdot \frac{\sum \alpha_i \cdot \phi(\frac{t-t_i}{\epsilon})}{\sum \alpha_i}.$$

Through our work we used different test functions to examplify $\mathcal{D}$-immersions upon two criterias: their positiveness and their symmetry with respect to the origin. In paricular, we expect positive test functions to respect convexity properties, in particular for the positive tangent cone, and symmetric test functions to respect the symmetries of the immersion. That leads us to consider five types of test functions: a basic gaussian-like test function (φ^1), a positive and symmetric one (φ^2), a non-positive and symmetric (φ^3), a positive and non-symmetric one (φ^4) and finally a non-positive and non-symmetric one (φ^5). Here is their expression in function of ϕ:

$$\begin{aligned}
\varphi^1 &= \phi(t),\\
\varphi^2 &= \frac{\phi(t-1)+\phi(t+1)}{2},\\
\varphi^3 &= \frac{2\phi(t-1)+\phi(t+1)}{3},\\
\varphi^4 &= -\phi(t-1)+3\phi(t)-\phi(t+1),\\
\varphi^5 &= 2\phi(t-1)-\phi(t+1).
\end{aligned}$$

6.2 Experimental setup

In order to estimate the direction of the tangent plane and the curvature at a given point of an approximated immersion in the plane we use a Maple-based program (see appendix A). We use the five test functions $\varphi^1 \ldots \varphi^5$ described previously to proceed with the tests. Three parameters are to be set to compute a test. The main one is the value of ϵ, the parameter relative to the approximation of the identity. We fixed three values for ϵ along our tests: 0.1, 0.05 and 0.01. Another parameter named translation sets the overlapping between the different bumps a test function can have. We set it to $\frac{9}{10}$ in order to have a small overlapping between the bumps. As a matter of fact, we observed that a small overlap allows a cleaner convolution between the test function and the immersion. The last parameter to set is the number of digits we want Maple to work with for numerical evaluations, although Maple tries to perform most of the evaluations formally. A greater number of digits oftenly leads to computational issues and we thus try to optimize the value of this parameter

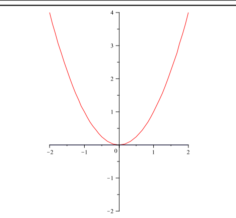
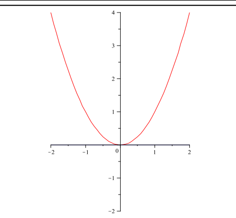
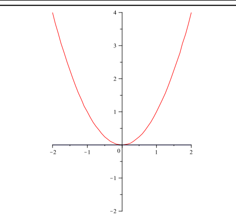
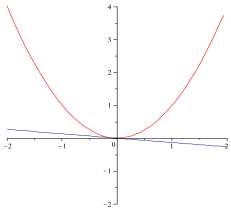
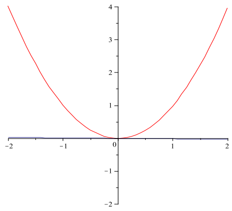
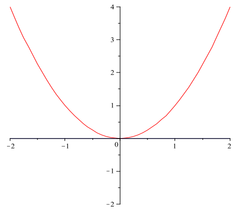
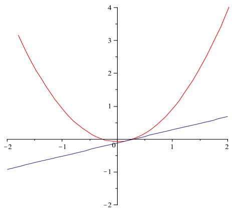
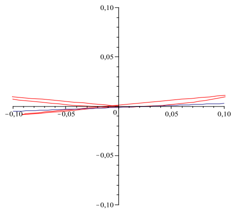
M_φ	$\epsilon = 0.1$	$\epsilon = 0.01$	$\epsilon = 0.001$
$\phi(t)$			
$\frac{2}{3}\phi(t) + \frac{1}{3}\phi(t+2)$			
$2\phi(t) - \phi(t+2)$			

Table 6.2: Mean manifolds with the mean tangent line for the parabola.

for each test. We fix it according to the value of the ϵ parameter: 5, 8 and 10 digits for $\epsilon = 0.1, 0.05, 0.01$ respectively.

Finally, we choose a parabola, a V-function and a cusp to be our experimental immersions. All three of them are $\mathcal{D}$-immersions since they are all graphs of functions. Both non-smooth immersions are representative of low order singularities in classical geometry, and are thus two interesting samples in discrete geometry. In particular, the V-function is the typical example of polygonal curves which have many applications in discrete modelling.

We try the three dimensional case (see appendix B) as well, meanwhile we encountered several issues when executing the Maple data-sheet. Although we managed to produce correct three dimensional test functions from the base function ϕ, we could not have Maple compute numerical values of the convolution. This is due to the lack of numerical methods for double integrals in Maple. Actually, Maple computes the double integral of the convolution by iterated integrals, requiring a formal integration followed by a numerical integration. Since ϕ has no simple primitive, Maple is not able to perform the formal integration and thus cannot evaluate or plot the required convolutions.

6.3 Tests on a parabola

The first example is a C^∞ submanifold of dimension 1 immersed in the plane; namely a parabola parameterized as the graph of the square function

	$\epsilon = 0.1$		$\epsilon = 0.05$		$\epsilon = 0.01$	
Test function	**Tang**	**Curv**	**Tang**	**Curv**	**Tang**	**Curv**
φ^1	0	2	0	2	0	2
φ^2	0	2	0	2	0	2
φ^3	-0.06	1.98	-0.03	1.99	-0.006	1.99
φ^4	0	2	0	1.99	0	1.99
φ^5	-0.54	1.36	-0.27	1.79	-0.054	1.99

Table 6.3: Tangent plane direction and curvature estimation for the parabola.

(see Table 6.2). Define:

$$M = \Big\{(t, t^2), t \in \mathbb{R}\Big\}.$$

Let U be an open of $\mathbb{R}$, define f such that:

$$f : \begin{cases} U & \longrightarrow M \quad , \\ t & \longmapsto (t, t^2) \quad . \end{cases}$$

The function f is an immersion on M and as being the graph of a function theorem 4.3 ensures that T_f is a $\mathcal{D}$-immersion. Now let's see how this is related to the classical theory. We define f_ϵ as before:

$$f_\epsilon : \begin{cases} U & \longrightarrow M_{\varphi_\epsilon} \quad , \\ x & \longmapsto (f * \varphi_\epsilon)(x) \quad . \end{cases}$$

For all φ_ϵ such that f_ϵ is an immersion, M_{φ_ϵ} is a mean submanifold of the plane. We can observe on Table 6.2 nice approximations for positive test functions, while non-positive test functions may generate some instabilities. Table 6.3 gives the estimations of the direction of the tangent plane $Tang$ and the curvature $Curv$ at parameter value 0 of the immersion, based on the five test functions listed previously.

$$Tang = \frac{y'(0)}{x'(0)} \qquad Curv = \frac{x'(0)y''(0) \ - \ x''(0)y'(0)}{(x'(0)^2 \ + \ y'(0)^2 \ )^{\frac{3}{2}}}$$

Notice that for all symmetric test functions (i.e, φ^1, φ^2, φ^4) the tangent plane and the curvature are well approximated. For non-symmetric test functions, both the tangent plane and the curvature converge to their original values when ϵ tends to 0.

6.4
Tests on a V-function

To test our theory on topological submanifolds, we study the graph of the absolute value function. We are interested in studying the unique singularity

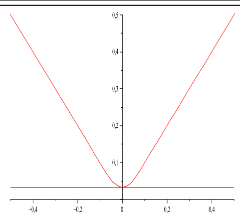
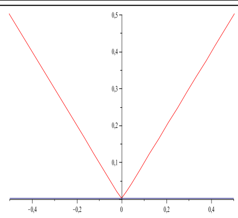
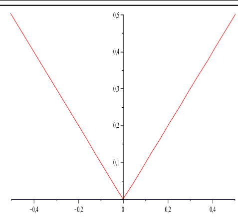
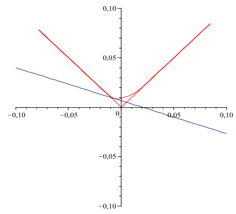
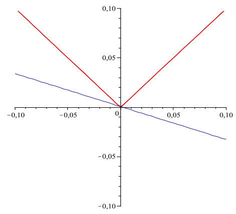
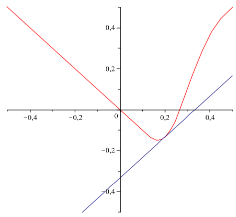
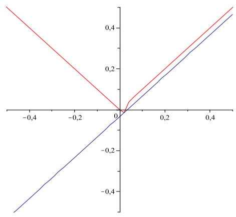
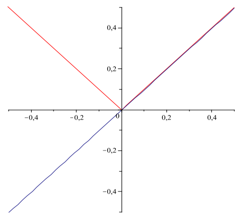
M_φ	$\epsilon = 0.1$	$\epsilon = 0.01$	$\epsilon = 0.001$
$\phi(t)$			
$\frac{2}{3}\phi(t) + \frac{1}{3}\phi(t+2)$			
$2\phi(t) - \phi(t+2)$			

Table 6.4: Mean manifolds with the mean tangent line for the V-shape.

of this submanifold which stands at the origin of the plane (see Table 6.4). Define:

$$N = \Big\{(t, |t|), t \in \mathbb{R}\Big\}.$$

Let V be an open of $\mathbb{R}$, define g such that:

$$g : \begin{cases} V & \longrightarrow N & , \\ t & \longmapsto (t, |t|) & . \end{cases}$$

The function g is a parameterization of N but it fails to be an immersion at $t = 0$. However by Theorem 4.3 T_g is a (non-trivial) $\mathcal{D}$-immersion. Now define g_ϵ as being the mean map of g:

$$g_\epsilon : \begin{cases} V & \longrightarrow N_{\varphi_\epsilon} & , \\ x & \longmapsto (g * \varphi_\epsilon)(x) & . \end{cases}$$

Therefore N_{φ_ϵ} is a mean submanifold of the plane for all $\epsilon > 0$, i.e, a tangent plane can be defined at any point. We can observe on Table 6.4 that non-symmetric test functions generate non horizontal tangent planes, even when ϵ is reduced. While symmetric test functions respect the symmetry of the right angle. We can actually prove this fact: Name s the unique singularity of N.

When ϵ tends to 0, we can compute the $\mathcal{D}$-tangent cone of T_g at s:

$$tg_{\mathcal{D}}(T_g, s) = \bigcup_{\varphi_\epsilon > 0} \Big(Acc\,\{D_s\ g_\epsilon\} \Big)(\mathbb{R}) \qquad .$$

Now looking at the $\mathcal{D}^+$-tangent cone of T_g at s, we have:

Proposition 6.1. *$tg^+_{\mathcal{D}}(T_g, s)$ respects the convexity of the submanifold N: all its elements are directions below N at s.*

Proof. Since N is a graph in the plane, it can parameterized by two functions, $x(t) = t$ and $y(t) = |t|$. Here $tg^+_{\mathcal{D}}(T_g, s)$ is below N when: $-1 \leq \dfrac{y'(0)}{x'(0)} \leq 1$. Theorem 4.3 ensures that $x'(0) = 1$:

$$x'(0) = -\int_{-\infty}^{\infty} t \cdot \varphi'_\epsilon(t)\ dt = -\Big[t \cdot \varphi_\epsilon(t)\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \varphi_\epsilon(t)\ dt = 1.$$

Now, the derivative of y at 0 is:

$$\begin{aligned} y'(0) &= -\int_{-\infty}^{\infty} |t| \cdot \varphi'_\epsilon(t)\ dt \\ &= -\int_{-\infty}^{0} (-t) \cdot \varphi'_\epsilon(t)\ dt - \int_{0}^{\infty} t \cdot \varphi'_\epsilon(t)\ dt \\ &= \Big[t \cdot \varphi_\epsilon(t)\Big]_{-\infty}^{0} - \int_{-\infty}^{0} \varphi_\epsilon(t)\ dt - \Big[t \cdot \varphi_\epsilon(t)\Big]_{0}^{\infty} + \int_{0}^{\infty} \varphi_\epsilon(t)\ dt \\ &= \int_{0}^{\infty} \varphi_\epsilon(t)\ dt - \int_{-\infty}^{0} \varphi_\epsilon(t)\ dt + \int_{-\infty}^{0} \varphi_\epsilon(t) - \int_{-\infty}^{0} \varphi_\epsilon(t) \\ &= x'(0) - 2 \cdot \int_{-\infty}^{0} \varphi_\epsilon(t) \end{aligned}$$

We obtain:

$$\frac{y'(0)}{x'(0)} = 1 - 2 \cdot \int_{-\infty}^{0} \varphi_\epsilon(t).$$

Since $\int_{-\infty}^{\infty} \varphi_\epsilon(t)\ dt = 1$, $\int_{-\infty}^{0} \varphi_\epsilon(t)\ dt \leq 1$. Moreover, since $\varphi_\epsilon \geq 0$, $1 \geq \int_{-\infty}^{0} \varphi_\epsilon(t)\ dt > 0$, we get:

$$-1 \leq 1 - 2 \cdot \int_{-\infty}^{0} \varphi_\epsilon(t) < 1.$$

And finally

$$-1 \leq \frac{y'(0)}{x'(0)} \leq 1.$$

□

Test function	$\epsilon = 0.1$ Tang	$\epsilon = 0.1$ Curv	$\epsilon = 0.05$ Tang	$\epsilon = 0.05$ Curv	$\epsilon = 0.01$ Tang	$\epsilon = 0.01$ Curv
φ^1	0	16.57	0	33.14	0	165.7
φ^2	0	0.23	0	0.46	0	2.33
φ^3	-0.33	0.19	-0.33	0.39	-0.33	1.99
φ^4	0	49.24	0	98.48	0	492.4
φ^5	-2.99	0.007	-2.99	0.014	-2.99	0.073

Table 6.5: Tangent plane direction and curvature estimation for the V-function.

Similarly to the previous example, Table 6.5 gives the estimations of the direction of the tangent plane and the curvature at parameter value 0 of the immersion. Here it can be observed that the curvature is inversely proportional to ϵ: when ϵ is divided by a certain amount, the curvature is multiplied by the same amount. This was an expected result since when ϵ tends to 0 the mean submanifold approximates the right angle with increasing precision linearly and thus the curvature rises linearly. Notice that here again symmetric test functions generate good approximations of the tangent plane.

6.5
Tests on a cusp

For the cusp, we only computed the estimations for the direction of the tangent plane and the curvature at parameter value 0 of the immersion (see Table 6.6). Once again due to the symmetry of the immersion, the direction of the tangent plane is well approximated when using symmetric test functions. The curvature explodes in absolute value when ϵ tends to 0: this behaviour corresponds to the non-linear structure of the cusp. Since it is a highly singular curve at its origin, approximating with convolution with a low valued ϵ results in a bad approximation of the curvature. When ϵ decreases the curvature rises rapidly. We can observe a lack of information around the point we focused on when testing with φ^3 since it has no symmetry. Moreover for test function φ^5, the weight on negative parts lead to a very low convergence and high numerical instability for the tangent plane, and flat approximations for the curvature. We

Test function	$\epsilon = 0.1$ Tang	$\epsilon = 0.1$ Curv	$\epsilon = 0.05$ Tang	$\epsilon = 0.05$ Curv	$\epsilon = 0.01$ Tang	$\epsilon = 0.01$ Curv
φ^1	0	36.29	0	102.6	0	1147.8
φ^2	0	-18.06	0	-51.09	0	-571.3
φ^3	-0.62	-11.04	-0.88	-21.57	-1.97	-52.95
φ^4	0	145.03	0	410.2	0	4586.1
φ^5	-5.60	-0.097	-7.93	-0.1	-17.7	-0.101

Table 6.6: Tangent plane direction and curvature estimation for the cusp.

would expect a similar result as property 6.1 for the positive tangent cone of the cusp, although in that case the tangent cone should be reduced to a single direction. Finally, it is trivial to see that the symmetric tangent cone for this immersion is actually reduced to the vertical direction.