5
Toward Geometric Properties of $\mathcal{D}$-Immersions

In the previous chapter we managed to generalize the concept of immersion by defining $\mathcal{D}$-immersion. The choice of approximations of the identity as test functions allows recovering of original parameterizations when parameter ϵ tends to 0, in the case of a $\mathcal{D}$-immersion associated with a smooth immersion. We will now exhibit geometric properties of $\mathcal{D}$-immersions. We want to know what kind of structure is mapped through a $\mathcal{D}$-immersion, and we will thus focus on a possible structuration of $D(U)$.

5.1
Structuring $D(U)$

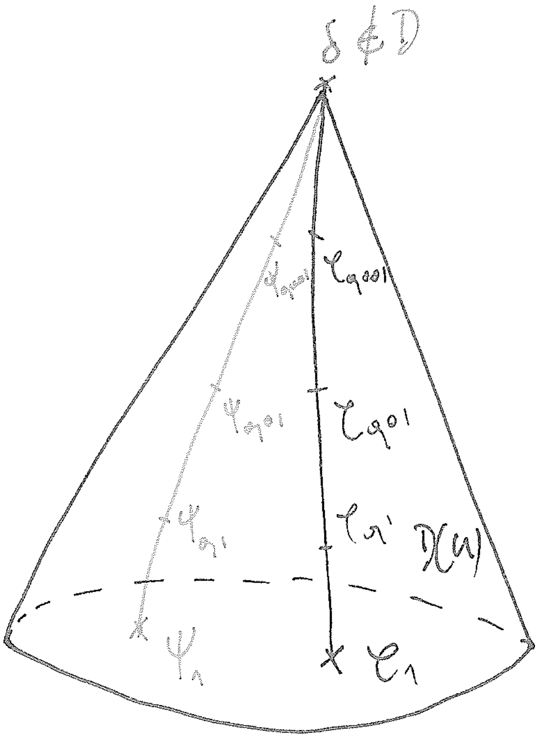

Figure 5.1: A structure on part of $D(U)$.

In this section, we will consider only the test functions $\phi \in D(\mathbb{R}^d)$ with

PUC-Rio - Certificação Digital Nº 0610731/CA

unit mass, denoting

$$D_*(U) = \{\phi \in D(U), \int_U \phi = 1\} \quad .$$

An immersion maps the local structure of $\mathbb{R}^d$ onto its image. Similarly, a $\mathcal{D}$-immersion maps a certain structure of $D_*(\mathbb{R}^d)$ onto its image. This structure is a collection of approximation curves: any $\phi \in D_*(\mathbb{R}^d)$ can be seen as an approximation of the identity $\phi = \tau_z \check{\varphi}_\epsilon$ for a certain z and ϵ (see Figure 5.1). For fixed ϵ, varying z, the test functions $\tau_z \check{\varphi}_\epsilon$ span a d-dimensional object in $D_*(\mathbb{R}^d)$. Reducing ϵ generates approximations of the identity at each point of this object. A $\mathcal{D}$-immersion maps this structure in $\mathbb{R}^n$: given a $\mathcal{D}$-immersion T, the image $\langle T, \phi \rangle \in \mathbb{R}^n$ of an arbitrary test function with unit mass ϕ is mapped onto an object $M_{\varphi_\epsilon} = T * \varphi_\epsilon(U)$ by $T * \varphi_\epsilon(z) = \langle T, \tau_z \check{\varphi}_\epsilon \rangle = \langle T, \phi \rangle$. If ϵ is small enough, but not zero, M_{φ_ϵ} is a smooth d-submanifold. Intuitively, part of the image of a $\mathcal{D}$-immersion can be seen as a collection of smooth submanifold in $\mathbb{R}^n$, eventually tending to an object in $\mathbb{R}^n$.

5.2 Compatible $\mathcal{D}$-immersions: change of parameters

In the classical context, we are able to characterize if two immersions define the same object. More precisely: $f\colon U_f \to M$ and $g\colon U_g \to M$ define the same object if $h = g^{-1} \circ f$ is a diffeomorphism. Applying h is called a change of parameters, and two immersions are compatible if they locally define the same object. Properties derived from immersions are geometric if they are invariant by change of parameters, otherwise they are merely analytical. We would like to state a similar characterization for $\mathcal{D}$-immersions. Since we use distributions as parameterizations, the change of parameters has to be done in spaces of test functions (see Figure 5.2).

Definition 5.1. *($\mathcal{D}$-change of parameters) We say that two $\mathcal{D}$-immersions $T \in D'(U)$ and $S \in D'(V)$ define the same object if:*

$$\exists h : U \to V \ \textit{and} \ \forall \varphi \in D_*(U),\ \exists \epsilon_0 > 0 \ \textit{and} \ \psi \in D_*(V) \quad \textit{such that}: \\ \forall \epsilon \in]0, \epsilon_0[, \forall u \in U, (T * \varphi_\epsilon)(u) = (S * \psi_\epsilon)(h(u)) \quad .$$

This definition may be restrictive, but it extends the notion of compatible immersions at least in the linear case, as stated in the next lemma:

Lemma 5.2. *If f and g are two compatible C^∞-immersions and $h = g^{-1} \circ f$ is linear, then T_f and T_g are compatible $\mathcal{D}$-immersions.*

PUC-Rio - Certificação Digital Nº 0610731/CA

PUC-Rio - Certificação Digital Nº 0610731/CA

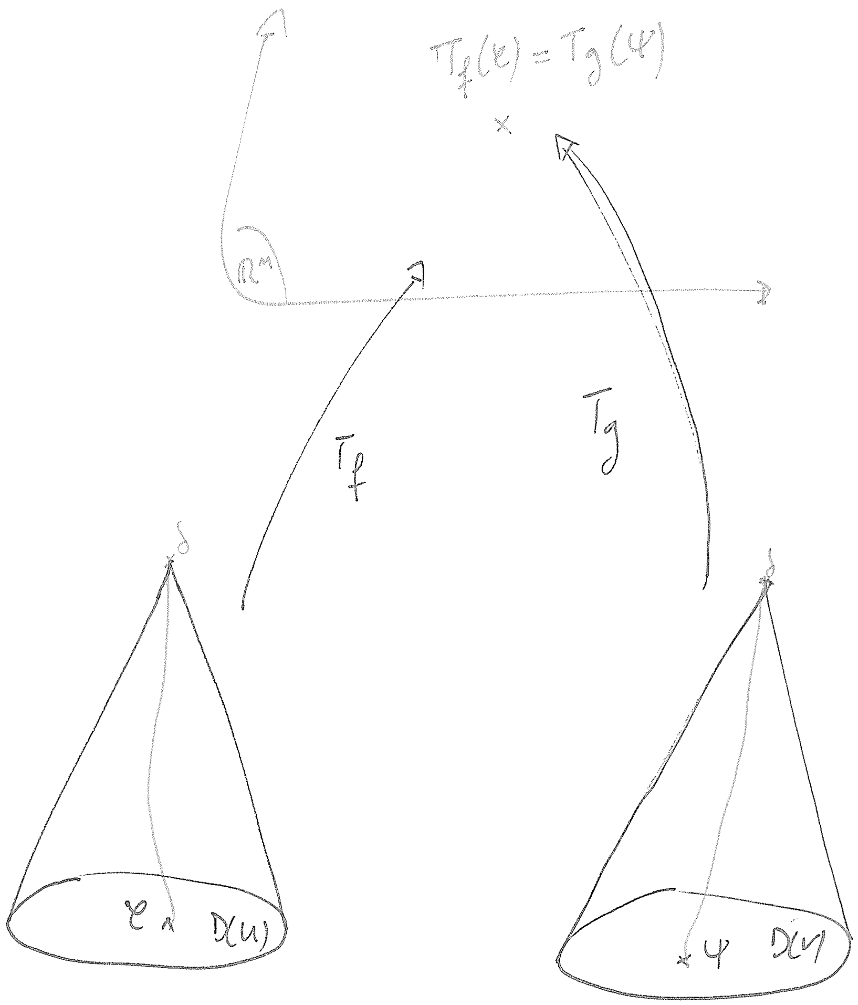

Figure 5.2: Compatible $\mathcal{D}$-immersions.

Proof. We know from Theorem 4.2 that T_f and T_g are $\mathcal{D}$-immersions. We have to check if they locally define the same object. Let $h = g^{-1} \circ f$ be the diffeomorphism mapping the domains U and V of f and g respectively.
Given $\varphi \in D_*(U)$, we want to determine $\psi \in D_*(V)$ such that $\langle T_f, \tau_u \check{\varphi}_\epsilon \rangle = \left\langle T_g, \tau_{h(u)} \check{\psi}_\epsilon \right\rangle$. We have:

$$
\begin{aligned}
\langle T_f, \tau_u \check{\varphi}_\epsilon \rangle &= \int_U f(x) \cdot \varphi_\epsilon(u - x) dx \\
&= \int_U g \ \circ \ g^{-1} \circ f(x) \cdot \frac{1}{\epsilon^d} \, \varphi \left(\frac{u - x}{\epsilon} \right) dx \\
&= \int_U g \ \circ \ h(x) \cdot \frac{1}{\epsilon^d} \, \varphi \left(\frac{u - x}{\epsilon} \right) dx \\
&= \int_V g \ \circ \ h \left(h^{-1}(y) \right) \cdot \frac{1}{\epsilon^d} \, \varphi \left(\frac{u - h^{-1}(y)}{\epsilon} \right) \cdot \left| \det J \left(h^{-1} \right) \right|^{-1} (y) \, dy \\
&= \int_V g(y) \cdot \varphi_\epsilon \left(u - h^{-1}(y) \right) \cdot \left| \det J \left(h^{-1} \right) \right|^{-1} (y) \, dy \quad .
\end{aligned}
$$

We can define, for a given ϵ:

$$\psi(y) = \varphi\left(h^{-1}(y)\right) \cdot \left|\det J\left(h^{-1}\right)\right|^{-1}(y)\,.$$

Observe that since h is a C^∞ diffeomorphism, ψ is C^∞ with support in V. Moreover, since we supposed that h is linear, we have that $J\left(h^{-1}\right)$ is a constant matrix and

$$\begin{aligned}\tau_{h(u)}\check{\psi}_\epsilon(y) &= \frac{1}{\epsilon^d}\,\psi\left(\frac{h(u)-y}{\epsilon}\right) = \frac{1}{\epsilon^d}\,\varphi\left(h^{-1}\left(\frac{h(u)-y}{\epsilon}\right)\right) \cdot \left|\det J\left(h^{-1}\right)\right|^{-1} \\ &= \varphi_\epsilon\left(u - h^{-1}(y)\right) \cdot \left|\det J\left(h^{-1}\right)\right|^{-1} \quad .\end{aligned}$$

Finally, $T_f * \varphi_\epsilon(u) = \langle T_f, \tau_u \check{\varphi}_\epsilon\rangle = \int_V g(y) \cdot \tau_{h(u)}\check{\psi}_\epsilon(y)dy = \left\langle T_g, \tau_{h(u)}\check{\psi}_\epsilon\right\rangle = T_g * \psi_\epsilon(h(u))$, with $\psi \in D_*(V)$. $\square$

From the last observation of the proof, the change of parameters works efficiently for the C^∞ case with linear domain mapping, but unfortunately the substitution formula does not work directly for other classes of functions. Here we face a delicate point of our proposal if we want to extend differential tools to the C^0 case.

5.3
The C^1 case

We were not able to define $\mathcal{D}$-change of parameters that extend directly C^0-subsitutions. However, it should be possible in the C^1 case. Indeed, we conjecture the $\mathcal{D}$-immersions associated to compatible C^1 immersions are $\mathcal{D}$-compatible. Follow elements of an eventual proof. Let f and g be two C^1 embeddings and name T_f and T_g their associated distributions. Given $\varphi \in C_0^\infty$ a test function, the C^1 substitution is $h = g^{-1} \circ f$. By the substitution formula of Theorem 5.2, we obtain $\psi = \varphi \circ h^{-1} |\det J(h^{-1})|^{-1}$. Since h is only C^1, ψ is only locally C^1. Hence ψ is not a test function. Given $\delta > 0$, there exists $\tilde{\psi}_\delta \in C_0^\infty$ such that:

$$\left|\int g \cdot \psi - \left\langle T_g, \tilde{\psi}_\delta\right\rangle\right| < \delta.$$

In that case, $\tilde{\psi}_\delta$ is a test function that may approximate the desired change of parameters for the C^1 case.

5.4
Tangent cones from $\mathcal{D}$-immersions

In order to study singular objects we have to be able to define approximations spaces upon singularities. Since tangent spaces cannot be defined ev-

PUC-Rio - Certificação Digital Nº 0610731/CA

erywhere on singular objects we propose a definition of tangent cone. We will see that this definition matches the definition of the common tangent space on regular objects.

5.4.1
Regular case

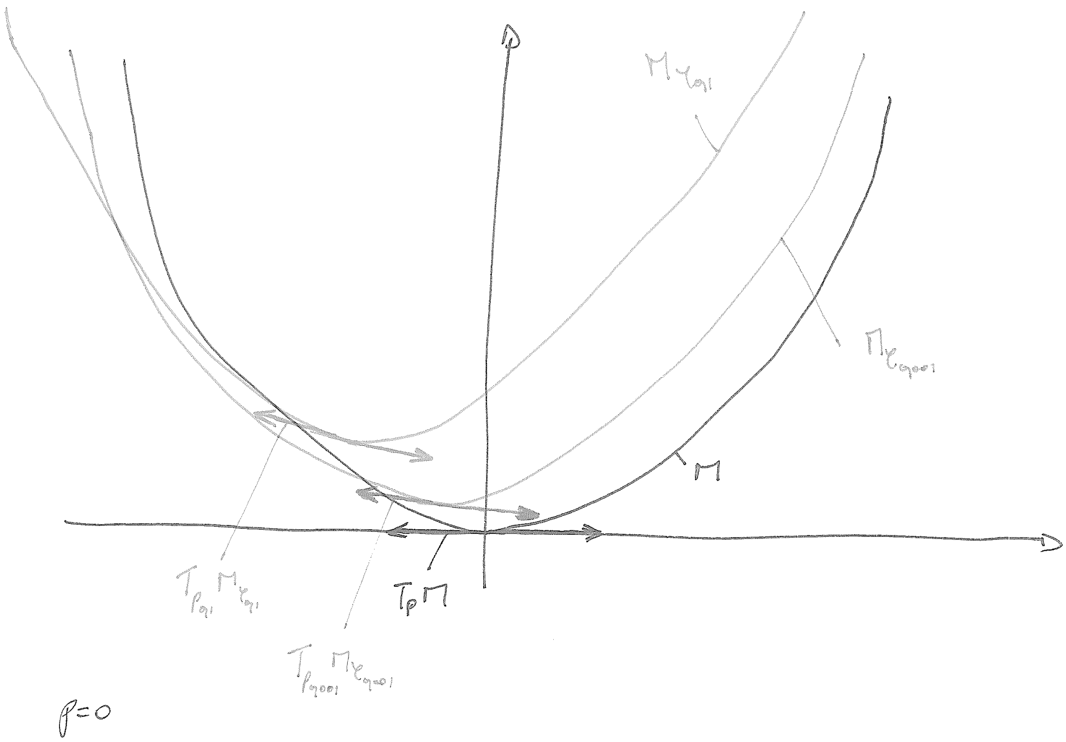

Figure 5.3: The tangent space at the images of the mean maps.

Consider a smooth immersion f and its associated $\mathcal{D}$-immersion T_f, and a fixed parameter $q \in U$. As recalled in Section 2.3, the tangent plane at $f(q)$ is the vector space $tg(f,q) = \mathcal{T}_{f(q)} f(U) = D_q\ f(\mathbb{R}^d)$. Since T_f is a $\mathcal{D}$-immersion, for $\varphi \in D_*(U)$ and for ϵ small enough, $f_\epsilon = T_f * \varphi_\epsilon$ is a smooth immersion. We can thus define $tg(f_\epsilon, q) = (D_q\ f_\epsilon)(\mathbb{R}^d)$. In the smooth case, we would expect $tg(f_\epsilon, q)$ to tend to $tg(f,q)$ (see Figure 5.3):

Proposition 5.3. *The derivative of $f_\epsilon = T_f * \varphi_\epsilon$ is :*

$$D_q f_\epsilon = D_q(f * \varphi_\epsilon) = (D_q f * \varphi_\epsilon).$$

Moreover: $\quad \lim_{\epsilon \to 0} D_q f_\epsilon = D_q f \quad .$

Proof. This is a direct consequence of Lemma 3.2 and Theorem 3.5. □

PUC-Rio - Certificação Digital Nº 0610731/CA

5.4.2
Singular case: classical approach

A simple definition for tangent cones on continuous objects can be stated as:

Definition 5.4. *Given a set $K \in \mathbb{R}^n$, we say that $w \in \mathbb{R}^n$ belongs to the tangent cone at s to K, denoted by $T(s,K)$, if there exists a sequence $(h_m)_m \in (\mathbb{R}^n)^{\mathbb{N}}$ where $h_m \neq 0$ and a sequence $(\lambda_m)_m \in (\mathbb{R})^{\mathbb{N}}$ where $\lambda_m > 0$, such that:*

$$\begin{cases} h_m \xrightarrow[m\to\infty]{} w \\ \lambda_m \xrightarrow[m\to\infty]{} 0 \end{cases} \quad \text{and} \quad \forall m, s + \lambda_m h_m \in M.$$

5.4.3
$\mathcal{D}$-tangent cone

Now, we intend to define a tangent cone directly from a $\mathcal{D}$-immersion. Following the regular case, the tangent cone of a $\mathcal{D}$-immersion T from parameter $q \in U$ would be the limit of $D_q T_\epsilon(\mathbb{R}^d)$, where $T_\epsilon = T * \varphi_\epsilon$ is a smooth immersion for small ϵ. This brute idea must overcome three delicate points: First, it would be a vector space convergence, and the tangent cone may not be a vector space. To overcome this, we can look at the function limit of $D_q T_\epsilon$. Second, for a general distribution, this may not converge to a function. We will thus look at the accumulation points instead of the limit. Last, this definition may depend of the approximation of the identity φ_ϵ used. Therefore we consider the union of the limits for all the approximations of the identity. This leads to the following definition:

Definition 5.5. *($\mathcal{D}$-Tangent Cone) Let T be a $\mathcal{D}$-immersion, $q \in U$ a fixed parameter. The $\mathcal{D}$-tangent cone of T at q denoted by $tg_{\mathcal{D}}(T,q)$, by:*

$$tg_{\mathcal{D}}(T,q) = \bigcup_{\varphi_\epsilon} \Big(Acc \left\{ D_q \; T_\epsilon \right\} \Big)(\mathbb{R}^d),$$

where Acc denote the set of accumulation points in the L^1 topology.

The $\mathcal{D}$-tangent cone can be restricted by applying conditions on φ_ϵ.

Definition 5.6. *($\mathcal{D}^+$-Tangent Cone) Let T be a $\mathcal{D}$-immersion, $q \in U$ a fixed parameter. The restricted $\mathcal{D}$-tangent cone of T at q denoted by $tg^+_{\mathcal{D}}(T,q)$, by:*

$$tg^+_{\mathcal{D}}(T,q) = \bigcup_{\varphi_\epsilon > 0} \Big(Acc \left\{ D_q \; T_\epsilon \right\} \Big)(\mathbb{R}^d) \qquad .$$

PUC-Rio - Certificação Digital Nº 0610731/CA

Similarly to this *positive tangent cone*, we can define the *symmetric tangent cone* by restricting the test functions φ_ϵ to be symmetric with respect to the origin.

Remark 5.7. *The $\mathcal{D}$-tangent cone is invariant by $\mathcal{D}$-change of parameters.*

Remark 5.8. *Proposition 5.3 ensures that, if f is an immersion, $tg_{\mathcal{D}}(T_f, q) = tg^{+}_{\mathcal{D}}(T_f, q) = tg(f, q)$, i.e. the $\mathcal{D}$-tangent cone extends the classical tangent cone.*

5.5
Intuitive proposal for $\mathcal{D}$-submanifold

The next step would be to combine $\mathcal{D}$-immersions in atlases to form $\mathcal{D}$-submanifolds, and to give an intrinsic definition for these objects. This section proposes a description of such objects in an informal way. We could define a $\mathcal{D}$-submanifolds $\mathcal{M}$ as a subset of $\mathbb{R}^n$ which is locally the limit of the images of $\mathcal{D}$-immersions:

$\mathcal{M} \subset \mathbb{R}^n$ is a $\mathcal{D}$-submanifold of dimension d if, for all point $x \in \mathcal{M}$, there exists :

- an open neighborhood V of x in $\mathbb{R}^n$,
- a compact K around V $(x \in V \subset K \subset \mathbb{R}^n)$,
- an open set U in $\mathbb{R}^d$,
- a $\mathcal{D}$-immersion T.

such that $\forall \varphi \in D_*(U), \quad \big((T * \varphi_\epsilon)(U)\big) \cap K \xrightarrow[\epsilon \to 0]{d_H} \mathcal{M} \cap K$, where $d_H(A, B) = \max\{\sup_{a \in A}(d(a, B)), \sup_{b \in B}(d(b, A)\}$ is the Hausdorff distance.
Moreover, if T and S satisfy the above criteria, then they must be compatible $\mathcal{D}$-immersions. This definition may be less restrictive if imposing only the existence of $\varphi \in D_*(U)$, instead of having the condition on all test function. The main challenge for this definition is to prove that a smooth submanifold is a $\mathcal{D}$-submanifold. This may be easier with the convergence in K, as suggested above, since the Hausdorff distance is reached on compacts.

PUC-Rio - Certificação Digital Nº 0610731/CA