4
Building Immersions with Distributions

The main objective of this work is to use distribution derivation on non-smooth immersions. Distributions are infinitely often differentiable objects, similarly to smooth parameterization. Therefore they naturally extend class conditions on immersions. In this chapter we set up a formulation for $\mathcal{D}$-immersions trying to preserve the main geometric properties of immersions.

4.1
Brute $\mathcal{D}$-parameterization: a first attempt

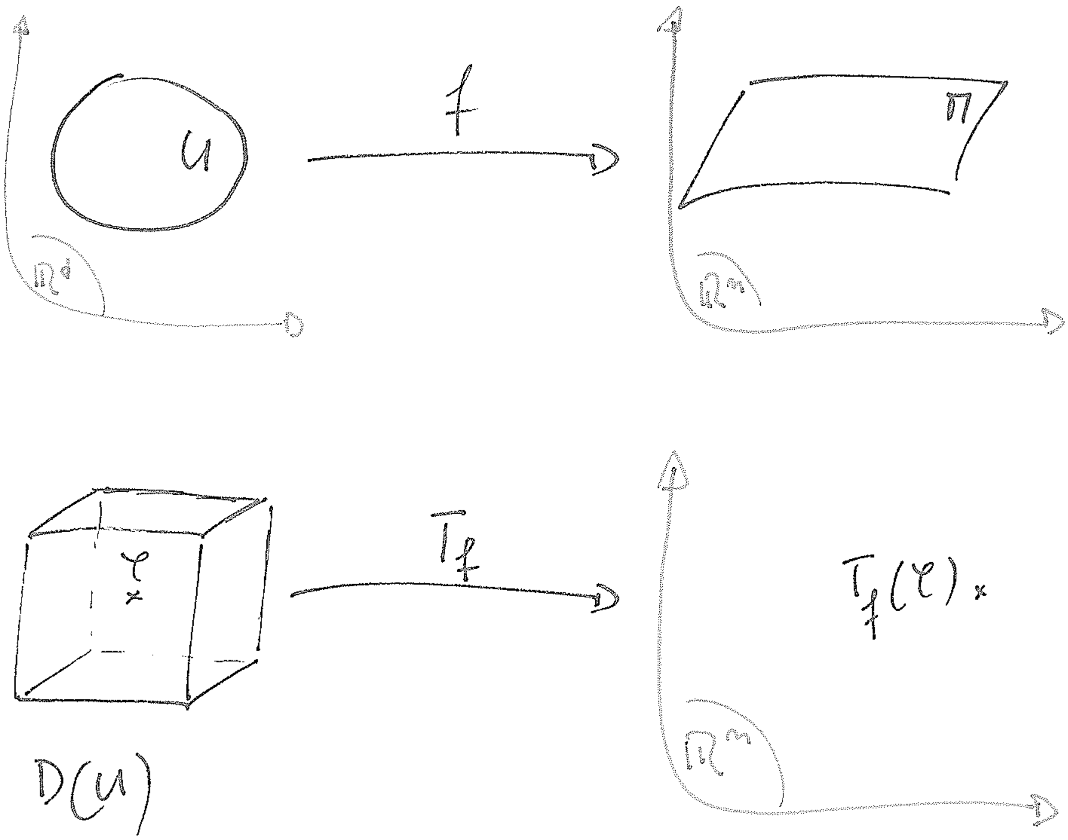

Figure 4.1: Parameterization directly from $D(\mathbb{R}^d)$.

There is a very direct way to substitute classical parameterizations by distributions. Distributions are defined on test functions spaces. Therefore, if we use a distribution T instead of a parameterization f, we change the parameter space from a subset U of $\mathbb{R}^d$ to $D(U)$. However test functions space $D(U)$ have infinite dimensions, and thus the submanifold parameterized on a test function space could have as many dimensions as the co-domain of the distribution used has (see Figure 4.1). This is clearly an undesirable fact. Another drawback concerns the derivative of our parameterization; we do not know how to interpret, in terms of tangent space, the derivation of T with respect to the space of test functions: $D_{\varphi_0}(T)$. As a direct use of distributions as parameterizations may not work mainly because of the non-finite dimension of the parameter space, we propose to structure differently the parameter spaces, in an approximation perspective.

4.2 Approximating by convolution

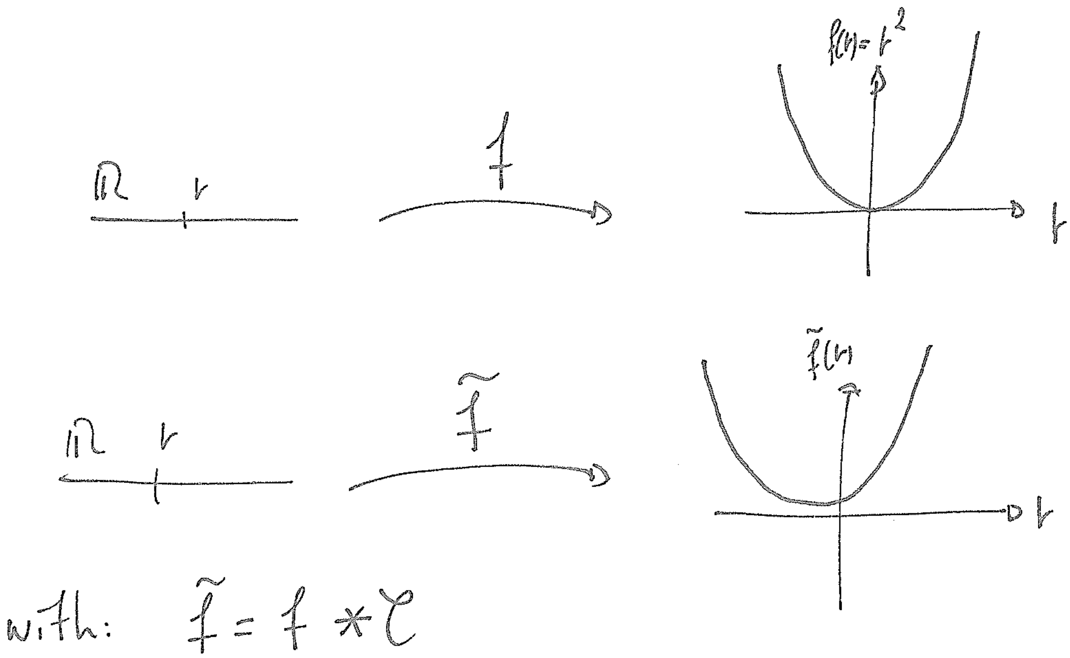

Figure 4.2: Convolution regularizes the parameterization.

On the one hand, defining differential properties from non-differential objects is often handled as an approximation problem, usually requiring convolution operations. On the other hand we saw in the previous chapter that applying a distribution and convolution product are related operations and that we could switch from one to another easily. Our approach lies in the regularization properties of the convolution product. Indeed, when computing the convolution product of a distribution with a test function, we obtain a C^∞ function (see Figure 4.2). Hence, convolution allows using distributions as smooth parameterizations. Moreover, it is possible to think of a more general parameterization not relying on an existing embedding: by taking an arbitrary distribution we can always generate a new parameterization by computing its convolution product with a test function. More specifically, let f be an immersion on $U \subset \mathbb{R}^d$, parameterizing its image $f(U) = M$ (M is a submanifold of dimension d in $\mathbb{R}^n$). Let T_f be the distribution associated to f, and φ be a test function on U. We define the function $\tilde{f}$ by:

$$\tilde{f} : \begin{cases} U & \longrightarrow \widetilde{M} \subset \mathbb{R}^n \\ x & \longmapsto \big(T_f * \varphi\big)(x) \end{cases}$$

Since the convolution averages functions, the function $\tilde{f}$ does not map U exactly on M, the image of the immersion f. Hence $\tilde{f}$ maps U on $\widetilde{M} = \tilde{f}(U)$, a mean set of M in $\mathbb{R}^n$. Hence $\tilde{f}$ parameterizes a geometrical object that corresponds to means of a classical submanifold $M = f(U)$ and those means depend on the test function φ used. In order to parameterize the original M, we need to choose the test function φ such that $\widetilde{M} = M$. This is in general not possible directly, but at the limit as in the regularization seen in Section 3.1.2. Formally, we can define a sequence of φ_ϵ of test functions such that $\big(T_f * \varphi_\epsilon\big)(U) \underset{\epsilon \to 0}{\longrightarrow} M$.

4.3
$\mathcal{D}$-immersions

In the formalization of this approach, we will try to preserve the geometric properties of the immersion f. We called the equivalent formulation for immersion $\mathcal{D}$-immersions (see Figure 4.3):

Definition 4.1. *($\mathcal{D}$-immersions) T is a $\mathcal{D}$-immersion if for all approximation of the identity φ_ϵ there exists $\epsilon_0 > 0$ such that for all $\epsilon \in]0, \epsilon_0[$, $T * \varphi_\epsilon$ is an immersion.*

To ensure that $\mathcal{D}$-immersions actually generalize classical immersions in the same way distributions generalize functions, we have to make sure that the distribution associated to a smooth immersion is actually a $\mathcal{D}$-immersion. This is done in the following theorem.

Theorem 4.2. *Let f be an immersion such that $f : U \to M$, where U is an open set of $\mathbb{R}^d$ such that $\overline{U}$ contains no singularity of f. If T_f is the distribution associated to f then T_f is a $\mathcal{D}$-immersion.*

Proof. Let φ_ϵ be an approximation of the identity, define f_ϵ such that:

$$f_\epsilon(x) = \big(T_f * \varphi_\epsilon\big)(x) = \int_U f(z) \cdot \varphi_\epsilon(x - z) dz = \big(f * \varphi_\epsilon\big)(x).$$

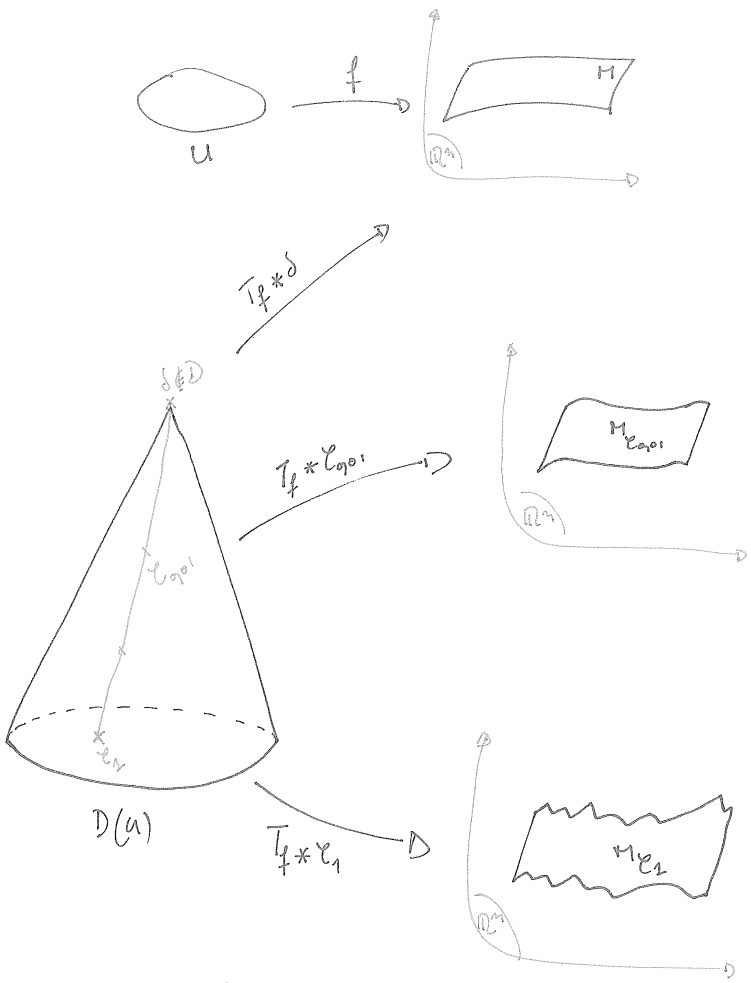

Figure 4.3: Image of a $\mathcal{D}$-immersion associated to an immersion.

We have to prove that for ϵ small enough f_ϵ is an immersion. For that, we will show that $D_x f_\epsilon$ has maximal rank for all $x \in U$ and $0 < \epsilon < \epsilon_0$. First observe that $D_x f_\epsilon$ is actually an approximation of $D_x f$:

$$D_x f_\epsilon : \begin{cases} \mathbb{R}^d & \longrightarrow \mathbb{R}^n \\ x & \longmapsto \big(Df * \varphi_\epsilon\big)(x) \end{cases} .$$

Since f is an immersion, $D_x f$ has at least one non-vanishing minor, i.e. there exists a $d \times d$ matrix $[D_x f]_d$ extracted from $D_x f$ such that $\det[D_x f]_d \neq 0$ for all $x \in U$. Denote by $[D_x f_\epsilon]_d$ the matrix extracted from $D_x f_\epsilon$ in the same way that $[D_x f]_d$ is extracted from $D_x f$.

The smaller ϵ, the closer $\det[D_x f_\epsilon]_d$ is from $\det[D_x f]_d$ and the further from 0. Formally, Theorem 3.5 ensures that $\det[D_x f_\epsilon]_d \underset{\epsilon\to 0}{\longrightarrow} \det[D_x f]_d$, since det is a continuous function:

$$\forall \alpha > 0,\ \exists \beta_\alpha > 0 \text{ such that } |\epsilon - 0| < \beta_\alpha \Rightarrow |\det[D_x f_\epsilon]_d - \det[D_x f]_d| < \alpha.$$

Choose α_0 to be:

$$\alpha_0 = \inf_{x \in U} \big(|\det[D_x f]_d| \big).$$

Since there are no singularities in $\overline{U}$, $\alpha_0 > 0$.
Since a ball centered in $\det[D_x f]_d$ of radius inferior to α_0 on the real line does not contain 0, for all $|\epsilon| < \beta_{\alpha_0}$ we have that $\det[D_x f_\epsilon]_d \neq 0$ and thus $D_x f_\epsilon$ has maximal rank. Concluding for all $x \in U$, exists $\beta_{\alpha_0} > 0$ such that for all $\epsilon < \beta_{\alpha_0}$, f_ϵ is an immersion. □

4.4
Graph of a function: $\mathcal{D}$-immersions from non-smooth immersions

In this section, we will prove that parameterizations of graph of functions, even if only L^1, are associated to $\mathcal{D}$-immersions. This allows using $\mathcal{D}$-immersions for a much wider class of objects.

Theorem 4.3. *Let $M \in \mathbb{R}^n$ be the graph of a function $u \in L^1 : \mathbb{R}^{n-1} \to \mathbb{R}$. Let U be an open of $\mathbb{R}^{n-1}$ and f be the parameterization of M such that:*

$$f : \begin{cases} U & \longrightarrow M \\ x & \longmapsto \big(f_1(x), \ldots, f_n(x)\big) \end{cases}$$

Where $\forall i \in \{1, \ldots, n-1\}$, $f_i : x \in \mathbb{R}^{n-1} \mapsto x_i \in \mathbb{R}$ and $f_n(x) = u(x)$.
The distribution associated to f is a $\mathcal{D}$-immersion.

Proof. Let φ_ϵ be an arbitrary approximation of the identity. We have to prove that the mean map of f is an immersion. Defines f_ϵ as being the mean map of f, $f_\epsilon = T_f * \varphi_\epsilon = f * \varphi_\epsilon$:

$$f_\epsilon \begin{cases} U & \longrightarrow M_{\varphi_\epsilon} \\ x & \longmapsto \Big(\big(f_1 * \varphi_\epsilon\big)(x), \ldots, \big(f_n * \varphi_\epsilon\big)(x)\Big) \end{cases}$$

We have to show that the rank of the Jacobian matrix is $n-1$. The Jacobian matrix of f_ϵ is:

$$\begin{pmatrix} \frac{\partial(f_1 * \varphi_\epsilon)}{\partial x_1} & \cdots & \frac{\partial(f_1 * \varphi_\epsilon)}{\partial x_{n-1}} \\ \vdots & & \vdots \\ \frac{\partial(f_n * \varphi_\epsilon)}{\partial x_1} & \cdots & \frac{\partial(f_n * \varphi_\epsilon)}{\partial x_{n-1}} \end{pmatrix}_{n \times \ n-1}.$$

$$\text{Since} \qquad \forall i \in \{1, \ldots, n-1\} \quad f_i(x) = x_i$$

$$\text{we have} \quad \frac{\partial(f_i * \varphi_\epsilon)}{\partial x_j} = \frac{\partial(f_i)}{\partial x_j} * \varphi_\epsilon = \frac{\partial(x_i)}{\partial x_j} * \varphi_\epsilon.$$

$$\frac{\partial(f_i * \varphi_\epsilon)}{\partial x_j} = \begin{cases} 1 * \varphi_\epsilon = \displaystyle\int 1 \cdot \varphi_\epsilon(x) dx = 1 & if \quad i = j \\ 0 * \varphi_\epsilon = 0 & if \quad i \neq j \end{cases} \qquad \forall (i,j) \in \{1, \ldots, n{-}1\}.$$

Thus the Jacobian matrix of f_ϵ is:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ f_n * \frac{\partial(\varphi_\epsilon)}{\partial x_1} & & \dots & & f_n * \frac{\partial(\varphi_\epsilon)}{\partial x_{n-1}} \end{pmatrix}_{n\times\ n-1} .$$

Hence the rank of the Jacobian matrix of f_ϵ is $n-1$, and consequently f_ϵ is an immersion. $\square$