3
Basics of Distribution Theory

To introduce some basics concepts of Distribution theory we first define the convolution product between two functions and then approximations of the identity. Then we define distributions and see how they generalize functions. The last section is dedicated to operations on distributions: the derivative of a distribution and the convolution product of a distribution and function. For further references and proofs see (Lebeau 1999) and see (Schwartz 1997) for more information on the invention of distributions.

3.1
Function approximations

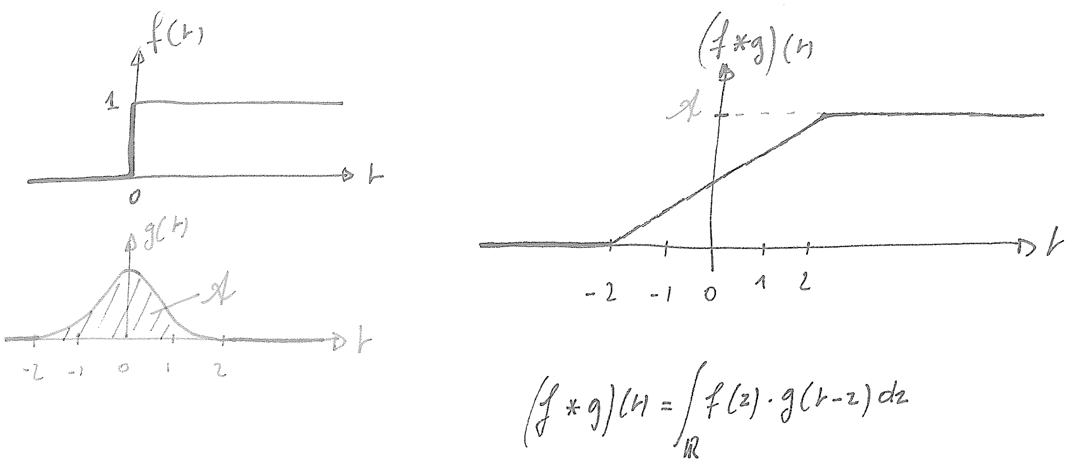

Figure 3.1: Convolution of a discontinuous function f with a smooth test function g.

3.1.1
Convolution

Let $L^1(\mathbb{R}^d)$ denote the normed vector space of integrable functions on $\mathbb{R}^d$, where $\mathbb{R}^d$ is equipped with its Lebesgue measure dx. Let f and g be two function of $L^1(\mathbb{R}^d)$.

Definition 3.1. *(Basic convolution) For $f, g \in L^1(\mathbb{R}^d)$, we call convolution product of f and g, denoted by $f * g$, the element of $L^1(\mathbb{R}^d)$ defined for almost* *every z by:*

$$(f * g)(x) = \int f(z)g(x-z)dz.$$

Figure 3.1 illustrates the regularization effect of the convolution.

Lemma 3.2. *Let $f \in L^1(\mathbb{R}^d)$, $g \in C^k(\mathbb{R}^d)$. If g admits limited partial derivatives $\partial^\alpha g$ for all multi-indexes α, $|\alpha| \le k$, then $f * g \in C^k(\mathbb{R}^d)$ and for $|\alpha| \le k$ we have:*

$$\partial^\alpha (f * g) = f * \partial^\alpha g.$$

3.1.2 Regularization

Let φ be a C^∞ real-valued function with support in the ball $\{\|t\| \le 1\}$ whose integral is equal to 1:

$$\int_{\mathbb{R}^d} \varphi(t)dt = 1.$$

Definition 3.3. *(Approximation of the identity) We call approximation of the identity the family of functions*

$$\Big(\varphi_\epsilon : t \longmapsto \epsilon^{-d}\varphi(t/\epsilon) \quad , \quad 0 < \epsilon \le 1\Big).$$

Note that the φ_ϵs are C^∞ functions with support in the ball $\{\|t\| \le \epsilon\}$ and of integral equals to 1, since

$$\int_{\mathbb{R}^d} \epsilon^{-d}\varphi(t/\epsilon)dt = \int_{\mathbb{R}^d} \varphi(t)dt = 1.$$

Lemma 3.4. *Let f be a continuous function with compact support on $\mathbb{R}^d$. The functions $f_\epsilon = f * \varphi_\epsilon$ belong to $C^\infty(\mathbb{R}^d)$, have a compact support and converge uniformly on $\mathbb{R}^d$ to f when ϵ tends to 0.*

Theorem 3.5. *For all $f \in L^1(\mathbb{R}^d)$, the functions $f_\epsilon = f * \varphi_\epsilon$ belong to $L^1 \cap C^\infty$, and converge in the L^1 norm to f when ϵ tends to 0.*

$$(f * \varphi_\epsilon) \underset{\epsilon \to 0}{\longrightarrow} f \quad in \quad L^1.$$

In particular, the space of $C^\infty(\mathbb{R}^d)$ functions with compact support is dense in $L^1(\mathbb{R}^d)$.

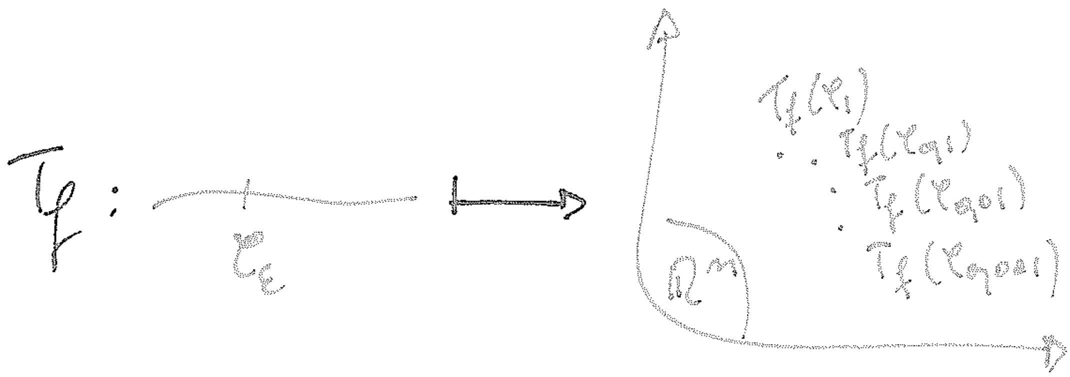

Figure 3.2: A distribution associates a point to a test function.

3.2 Distributions

For K compact of $\mathbb{R}^d$, we denote by C_K^∞ the space of C^∞ functions of $\mathbb{R}^n$ with support included in K. Let Ω be an open of $\mathbb{R}^d$. $C_0^\infty(\Omega)$ is the union of C_K^∞'s where K is a compact in Ω. The elements of $C_0^\infty(\Omega)$, i.e. infinitely often differentiable functions with compact support in Ω, are called *test functions.*

Definition 3.6. *(Distributions) A distribution on Ω is a linear form T of $C_0^\infty(\Omega)$*

$$\varphi \longmapsto \langle T, \varphi \rangle \in \mathbb{R} \qquad \varphi \in C_0^\infty(\Omega).$$

which satisfies the following property: for all compact K in Ω, there exists an integer p and a constant C such that

$$\forall \varphi \in C_K^\infty \qquad |\langle T, \varphi \rangle| \leq C \sup_{\substack{|\alpha| \leq p \\ x \in K}} |\partial^\alpha \varphi(t)|. \tag{3-1}$$

We denote by $\mathcal{D}'(\Omega)$ the space of distributions on Ω (see Figure 3.2). It is a vector space. When the integer p can be chosen independently from K, we say that the order of the distribution T is finite, and the smallest possible value of p is called the order of T.

Distribution are "generalized functions". Let $L^1_{loc}(\Omega)$ be the space of functions locally integrable on Ω. An element of $L^1_{loc}(\Omega)$ is the data of a Lebesgue-measurable function f on Ω, satisfying $\int_K |f(x)| dx < \infty$ for all compact $K \in \Omega$: two such functions are identified if and only if $f(x) = g(x)$ almost everywhere. We write T_f, the distribution associated to an element $f \in L^1_{loc}(\Omega)$ i.e.

$$\langle T_f, \varphi \rangle = \int_\Omega f(x) \varphi(x) dx \qquad \forall \varphi \in C_K^\infty.$$

From definition 3.6, we have $|\langle T_f, \varphi \rangle| \leq C \sup_{x \in K} |\varphi(x)|$ with $C = \displaystyle\int_K |f(x)| dx$, so the regularity condition (3-1) is satisfied for $p = 0$. The next lemma identifies $L^1_{loc}(\Omega)$ with a subspace of $\mathcal{D}'(\Omega)$.

Lemma 3.7. *Let f and g be two functions locally integrable on Ω. The following properties are equivalent:*

– $f(x) = g(x)$ *almost everywhere.*
– $\displaystyle\int f(x)\varphi(x)dx = \int g(x)\varphi(x)dx$ *for all $\varphi \in C_0^\infty(\Omega)$, i.e. $T_f = T_g$.*

Example 3.8. *Dirac's distribution at $a \in \mathbb{R}^d$, δ_a is defined by*

$$\langle \delta_a, \varphi \rangle = \varphi(a).$$

It is a distribution of order 0 *on $\mathbb{R}^d$. If $\chi_\epsilon(x) = \epsilon^{-1}\chi(x/\epsilon)$ is an approximation of the identity, the χ_ϵ's converge point-wise in $\mathcal{D}'(\mathbb{R}^d)$ to δ_0 since*

$$\langle \chi_\epsilon, \varphi \rangle = \int \varphi(x)\chi_\epsilon(x)dx = \big(\varphi * \check{\chi}_\epsilon\big)(0).$$

where $\check{\chi}_\epsilon(x) = \chi_\epsilon(-x) = \epsilon^{-1}\chi(-x/\epsilon)$ is also an approximation of the identity, hence by the lemma 3.4:

$$\lim_{\epsilon \to 0} \langle \chi_\epsilon, \varphi \rangle = \varphi(0) = \langle \delta_0, \varphi \rangle .$$

Example 3.9. *Another example of distribution, but of order* > 0, *is d_a defined for $a \in \mathbb{R}$ by*

$$\langle d_a, \varphi \rangle = \varphi'(a).$$

3.3
Operations on distributions

Definition 3.10. *(Derivation) The partial derivatives $\dfrac{\partial T}{\partial x_i}$ of a distribution $T \in \mathcal{D}'(\Omega)$ are the distributions on Ω defined by:*

$$\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle = \left\langle T, -\frac{\partial \varphi}{\partial x_i} \right\rangle \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Hence for $T \in D'(\Omega), \varphi \in C_0^\infty(\Omega)$

$$\left\langle \frac{\partial}{\partial x_i}\left(\frac{\partial T}{\partial x_j}\right), \varphi \right\rangle = \left\langle \frac{\partial T}{\partial x_j}, \frac{\partial \varphi}{\partial x_i} \right\rangle = \left\langle T, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle .$$

By Schwartz's lemma: $\dfrac{\partial}{\partial x_i}\left(\dfrac{\partial T}{\partial x_j}\right) = \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial T}{\partial x_i}\right)$. Therefore the order of derivation does not affect the result of a successive derivation of a distribution, and for $\alpha \in \mathbb{N}^n$ multi-index, we have:

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle .$$

Besides, if $T = f(x)$ is a C^1 function, Definition 3.10 means that for all $\varphi \in C_0^\infty$ we have:

$$\left\langle \frac{\partial T}{\partial x_j}, \varphi \right\rangle = \left\langle T, \frac{\partial \varphi}{\partial x_j} \right\rangle = -\int f(x) \frac{\partial \varphi}{\partial x_j}(x) dx = \int \frac{\partial f}{\partial x_j}(x) \varphi(x) dx.$$

Hence the distributional derivative, $\dfrac{\partial T_f}{\partial x_j}$ of T_f is the distribution associated to the usual derivative $\dfrac{\partial f}{\partial x_j}(x)$ for f of class C^1: $\frac{\partial T_f}{\partial x_j} = \frac{T \cdot \partial f}{\partial x_j}$.

Example 3.11. *Let $H(t)$ be Heaviside's function defined for $t \in \mathbb{R}$ by*

$$\begin{cases} H(t) = 1 & for\ t \geq 1 \\ H(t) = 0 & for\ t < 0 \end{cases}$$

$H \in L^1_{loc}(\mathbb{R})$ and is thus associated to a distribution. Computing its distributional derivative we obtain:

$$\langle T'_H, \varphi \rangle = -\langle T_H, \varphi' \rangle = -\int_0^\infty \varphi'(t) dt = \varphi(0) = \langle \delta_0, \varphi \rangle .$$

Hence:

$$T'_H = \delta_0.$$

Example 3.12. *Let $\delta_a(t)$ be Dirac's distribution defined for $t \in \mathbb{R}$. We can check that $\delta'_a = -d_a$:*

$$\langle \delta'_a, \varphi \rangle = -\langle \delta_a, \varphi' \rangle = -\varphi'(a).$$

Theorem and Definition 3.13. *(Substitution formula) Let Ω_1 and Ω_2 be two open subsets of $\mathbb{R}^d$ and $\phi : \Omega_1 \to \Omega_2$ a C^∞ diffeomorphism. For $T \in D'(\Omega_2)$, the formula:*

$$\forall \varphi \in C_0^\infty(\Omega_1), \quad \langle T \circ \phi, \varphi \rangle = \langle T, \psi \rangle \quad with \quad \psi(y) = \frac{\varphi\big(\phi^{-1}(y)\big)}{|\det J\big(\phi^{-1}(y)\big)|}.$$

defines a distribution on Ω_1, called inverse image of T by the change of parameter ϕ.

The above definitions match the usual formula for distributions associated to a function in L^1.

Now, we aim at calculating the convolution product of distributions, we first define the convolution product of a distribution and a test function. Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$. We define a function of x by setting:

$$\big(T * \varphi\big)(x) = \langle T, \tau_x \check{\varphi} \rangle \,. \tag{3-2}$$

where $\tau_x \check{\varphi} \in C_0^\infty(\mathbb{R}^d)$ is the function $z \mapsto \check{\varphi}(z - x) = \varphi(x - z)$. For $T = f(x) \in L^1_{loc}$, the formula above is equivalent to the usual definition $\big(f * \varphi\big)(x) = \displaystyle\int f(z)\varphi(x - z)dz$. Observe that $T * \varphi$ is always a C^∞ function.

Proposition 3.14. *We denote by $T * \varphi$ the convolution product of T and φ. It is a C^∞ function on $\mathbb{R}^d$ satisfying for all α:*

$$\partial^\alpha (T * \varphi) = T * \partial^\alpha \varphi.$$

Note that approximations of the identity also regularize distributions:

Proposition 3.15. *If φ_ϵ is an approximation of the identity, then we have the convergence in D': $T * \varphi_\epsilon \underset{\epsilon \to 0}{\longrightarrow} T$, i.e.:*

$$\forall \phi \in C_0^\infty(\Omega_1), \langle T * \varphi_\epsilon, \phi \rangle = \langle T, \phi * \check{\varphi}_\epsilon \rangle \underset{\epsilon \to 0}{\longrightarrow} \langle T, \phi \rangle \ \textit{in } \mathbb{R}.$$