2
Basics of Differential Geometry

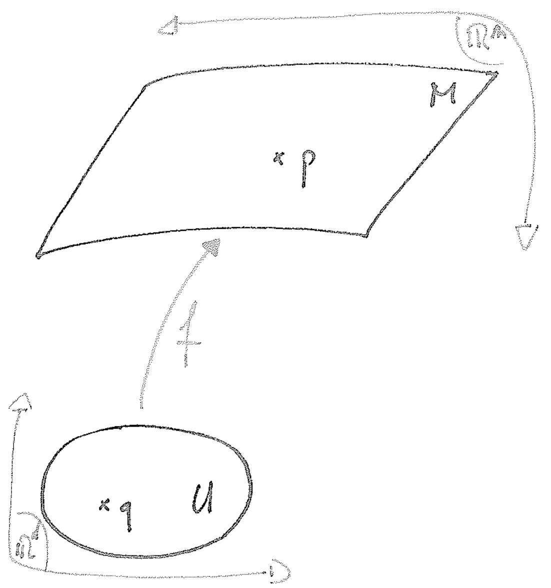

Figure 2.1: An immersion on a submanifold.

In this chapter we define some of the most basic tools of differentiable geometry. We begin by defining immersions to parameterize differential submanifolds. We finally state some useful theorems and definitions that we will need along our work. Along this dissertation we will use the word "smooth" to design C^∞ objects. Further references on differential geometry may be found in (do Carmo 2005).

2.1
Differential immersions and submanifolds

Definition 2.1. *(Immersion) Let U be an open set of $\mathbb{R}^d$. The function $f : U \to \mathbb{R}^n$ is called immersion if for all $x \in U$, f is differentiable and the rank of $D_x f$ is d.*

The notion of submanifold is an important concept in modern geometry since it allows complex structures to be expressed in terms of relatively well-understood properties of simpler spaces such as the Euclidian space $\mathbb{R}^d$. Every point of a submanifold has a neighborhood diffeomorphic to the Euclidian space. It is easier to work on a submanifold than on some unstructured geometrical object. If the local maps are compatible, it is possible to use calculus on a differential submanifold, in particular to define a tangent space.

Definition 2.2. *(Submanifold) $M \in \mathbb{R}^n$ is a* regular submanifold *of dimension* d *if for all $p \in M$ exists a neighborhood $V \subset \mathbb{R}^n$ and a map $f : U \to V \cap M$ of an open set $U \subset \mathbb{R}^d$ $(d < n)$ onto $V \cap M$ such that:*

- *f is differentiable.*
- *f is a homeomorphism.*
- *$\forall q \in U$, $D_q f : \mathbb{R}^d \to \mathbb{R}^n$ is one-to-one.*

In such case, f is called a parameterization of M around p.

Remark 2.3. *Every submanifold is locally the image of an immersion (see Figure 2.1).*

This remark is a direct consequence of the definition of a regular submanifold.

Definition 2.4. *(Embedding) Let f be an immersion. If f is a homeomorphism then f is called an embedding.*

Theorem 2.5. *The image of an embedding is a submanifold.*

Theorem 2.6. *Let U be an open set of $\mathbb{R}^d$. Let $f : U \to \mathbb{R}^n$ be an immersion. For all $x \in U$, there exists a neighborhood W of x in U such that $f|_W : W \to \mathbb{R}^n$ is an embedding.*

This last theorem permits us to speak about immersed submanifold, i.e. submanifolds which are the images of injective immersions.

2.2 Parameter independence

One of differential geometry main objectives is to study local properties of regular submanifolds. Indeed, according to the definition, local coordinate systems exist in the neighborhood of each point p of a regular submanifold. It is thus possible to define local properties of the submanifold according to these coordinates. For example we can define differentiability at a point $p \in M$. If f is a function from M to another submanifold, an intuitive way to define the differentiability of f on M is to choose a coordinate system in the neighborhood of p and say that f is differentiable on M if its expression in the coordinates of the chosen neighborhood system is a differential map. But a point p of a

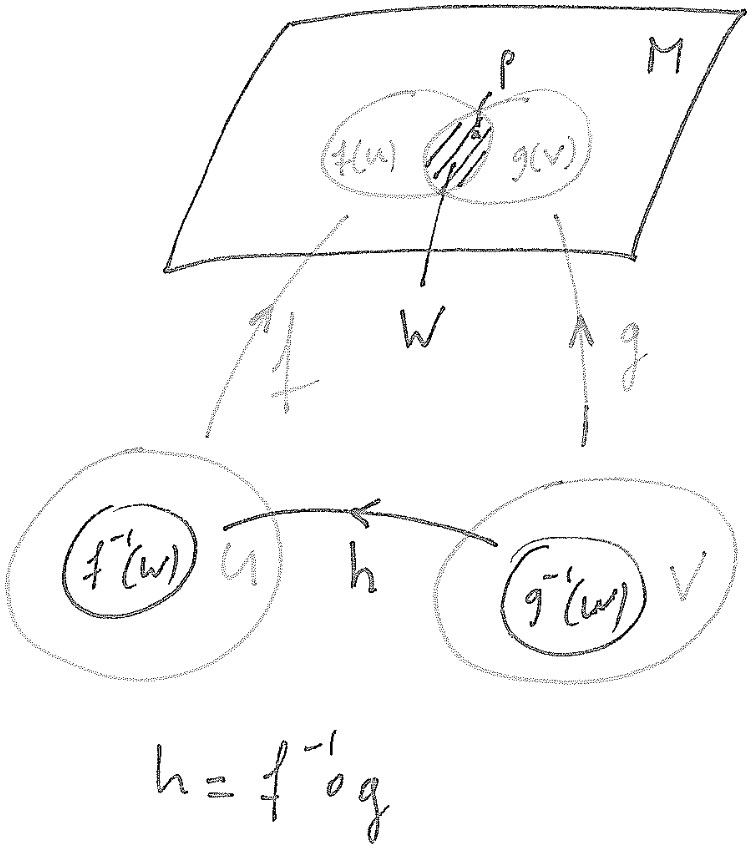

Figure 2.2: Change of parameterization.

regular submanifold belongs to various coordinate neighborhoods and we do not want our definition of differentiability to depend on the chosen coordinate system. Hence in order to define a local property such as differentiability on a regular submanifold geometrically we have to make sure that it is independent of the coordinate system chosen. The next proposition shows that the change of parameters preserves the differential structures of the submanifold.

Proposition 2.7. *Given $p \in M$, M submanifold, let f and g be two parameterizations of M such that:*

$$\begin{cases} f : U \longrightarrow M \\ g : V \longrightarrow M \end{cases} \qquad p \in f(U) \cap g(V) = W.$$

Define $h = (f^{-1} \circ g) : g^{-1}(W) \to f^{-1}(W)$. h is a diffeomorphism (see Figure 2.2).

2.3 Tangent space

Tangent spaces are the best linear approximation of the submanifold. Once tangent spaces have been defined it is possible to define vector fields on

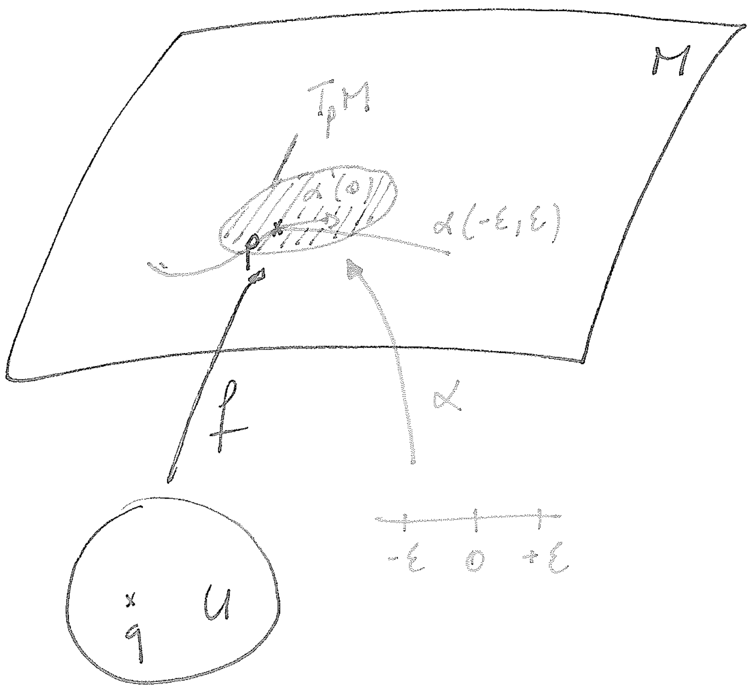

Figure 2.3: Tangent space of a submanifold at a regular point.

a submanifold and, further on, differential equations. The tangent space at a point p of a regular submanifold is a real vector space which intuitively contains all the possible directions in which one can pass through p (see Figure 2.3).

Definition 2.8. *A vector is tangent at p to M if it is the tangent vector $\alpha'(0)$ of a differentiable parameterized curve:*

$$\alpha : (\epsilon; \epsilon) \longrightarrow M \qquad \textit{with} \qquad \alpha(0) = p.$$

In order to define tangent spaces properly, we have to make sure that they do not depend on the parameterization used to parameterize the submanifold, this will ensured by the next proposition.

Proposition 2.9. *Let f be a parameterization of an open set of M and q such that $f(q) = p$.*

The set of tangent vectors at a point $p \in M$ is equal to $D_q f(\mathbb{R}^d)$.

Proof. ($\subset$) If w is a tangent vector at $f(q) = p$ we have $w = \alpha'(0)$ where $\alpha : (-\epsilon; \epsilon) \to M$ and $f(q) = \alpha(0)$.

Define:

$$\beta = f^{-1} \circ \alpha.$$

We have:

$$\beta'(0) = D(f^{-1} \circ \alpha)_0 = (Df)_q^{-1} \circ \alpha'(0).$$

Hence:

$$D_q f(\beta'(0)) = \alpha'(0) = w,$$

and $w \in D_q f(\mathbb{R}^d)$.

($\supset$) If $w = D_q f(v),\ v \in \mathbb{R}^d$. Define: $\gamma(t) = t.v + q$ and $\alpha = f \circ \gamma$, we have:

$$\alpha'(0) = D(f \circ \gamma)_0 = D_q f \circ \gamma'(0) = D_q f(v) = w.$$

Hence w is a tangent vector to the parameterized curve α and therefore is a tangent vector at the point $p \in M$.

$\square$

By the previous proposition, the space $D_q f(\mathbb{R}^d)$ can be calculated from an immersion f, but it does not depend on a particular choice for f. This space will be called the *tangent space* at p to M, and denoted $T_p M$.