1 Introduction

Geometry, one of the oldest kinds of sciences, was first recorded in the ancient Mesopotamia when men needed to measure the variations of the tide of a river. It was also used by the Egyptians, who studied shapes, and then first put into an axiomatic form by Euclides in the third century B.C. In the eighteenth century the study of intrinsic structure of geometrical objects made great advances through the work of Euler and Gauß. Gauß' Egregium Theorem states a way for computing the curvature of a surface without considering the ambient space in which the surface lies. In modern terms, this type of surface would be called a manifold. With the emergence of infinitesimal geometry and topology, a common way to describe particular manifolds requires the concept of immersion. In particular we are interested along this work in extending differential tools on immersions to singular geometrical configurations. It is a challenge for mathematicians to study these singular objects. In particular, Geometric Measure Theory (GMT) is a generalization of differential geometry through measure theory. It was mainly created by Federer (Federer 1996, Morgan 2000) to deal with maps and surfaces that are not necessarily smooth. GMT uses tools similar to distribution theory defining rectifiable sets as currents. Integral geometry (Santaló 1953, Langevin 2006) is another way to deduce geometric invariants without differentiation. It has its origin in the theory of geometrical probabilities.

This work is a tentative, among many others to develop a simple formalism to study singular objects. We choose to use distribution theory to extend differential geometry tools, since distributions already extend functions and measures. More specifically, we want to develop a coherent structure that allows substituting classical parameterizations of a differential submanifolds by distributions. Distributions were invented by Schwartz at the end of 1944 to generalize the notion of function. At the time it was a challenge to be able to define the derivative of any function at any point. By generalizing functions with infinitely often differentiable objects, Schwartz' discovery allowed solving many differentials problems, since a distributional derivative always exists, in contrast with the usual derivative. A similar generalization occurred in the history of mathematics when rational numbers were generalized by real numbers in order to solve the square root problem. In that context another generalization eventually emerged later because negative real numbers did not have any polynomial root, the creation of complex numbers solved the problem. The ergonomic aspect of distributions may be the key to unite differential and singular geometry.

Along this work we sketch a formalization mixing distributions and immersions. As distributions are can be infinitely derivable, they are the natural candidates to substitute classical parameterizations. We define $\mathcal{D}$-immersions, a generalization of immersions in the sense that the distribution associated to an immersion is a $\mathcal{D}$-immersion. Besides, we observe that graphs of L^1 functions are $\mathcal{D}$-immersions which motivated us to keep that definition and look further. This would be a first step toward another distributional extension of submanifolds. In particular, we study change of parameterizations through $\mathcal{D}$-immersions in this direction. Similarly, we propose a derivation of tangent cones to the local image of a $\mathcal{D}$-immersion, which match one of the usual tangent cone for images of smooth immersions.

This work is organized as follows. We first introduce basic concepts of differential geometry and distribution theory (Chapters I and II). We then define $\mathcal{D}$-immersions theory (Chapter III) and study some relations with immersions as manifold parameterization (Chapter IV). Finally we show some examples to illustrate this formulation and applications to geometric approximations (Chapter V).