4
Models for the contract analysis

The assessment of the contract proposed is carried out only over the contracts that are optimal for the Buyer-Supplier dyad and, also, viable for both companies. Those contracts are defined by their triples of parameter values that maximize the dyad's expected profit and make companies better off in terms of the expected profit. Since that the dyad's expected profit is assumed to be just the sum of Buyer's and Supplier's expected profits, the procedure to determine optimal and viable contracts involves solving the medium-term optimal capacity problem for each company under the contract conditions. To determine the viability for a given contract in each company, the individual optimal expected profits under the independent planning (i.e. without contract) are required. As already mentioned, the maximum expected profit of each company under independent planning will also provide means for assessing the gain that a given contract can bring to each company, i.e. how the contract splits the total gain between them. Also, to determine the gain in the companies' joint performance derived from the contract, the dyad's maximum expected profit under central planning has to be obtained. In the centralized planning system's capacity problem, the companies act as a single producer with two serval units coordinated by the central planning entity who decides on both capacities.

In the sequel, the formulation and solution for the companies' capacity problems are presented in Section 4.1 under the contract situation and the two benchmarks considered, namely: *independent planning* and *central planning*. The formulation of these problems is based in the modeling assumptions exposed in Section 3.5.1. The derivations and proofs are in Appendix C. Section 4.2 makes a general appraisal of the generic contract proposed in terms of the relation between the companies' optimal capacities, under and without the contract. And, Section 4.3 presents the maximization problem to be solved to find optimal contracts, which is here called *contract problem*, and the contract optimization procedure that will be applied.

4.1 The companies' capacity problems

A company decides on its medium-term production capacity taking into account the effect that this decision would have on its expected profit. Assuming that the medium-term capacity cost-function is perfectly known, the expected profit of the decision can be considered as the expected operational (short-term) profit minus the capacity cost. In the short-term, the production capacity is fixed and the operational profit for each company is given in terms of revenue from the sale of its product and the operational expenses, which include costs of inputs and variable production costs. The fixed costs can only modify the value of the profit by a constant and, then, they will not be considered. However, at the instant of making the medium-term capacity decision, each company knows only probabilistically the market demand for its product. So, assuming an expected profit maximizing behavior for the companies' management, the capacity problem is defined, here, as a bounded maximization problem.

For the manufacturing setting described in the previous chapter, the companies' products and inputs are traded in the market at prices denoted by p_p, p_m and p_{rm}, respectively, for the product, material and raw material. Denoting the variable production cost to produce the product and the material, respectively, by v_p and v_m, the Buyer's production margin (π_B), when he buys the material in the spot market, is given by $\pi_B = p_p - p_m - v_p$, while Supplier's production margin (π_S), when she sells the material in the market, is given by $\pi_S = p_m - p_{rm} - v_m$. It is assumed the markets are "viable" for both companies, that is, π_B and π_S are considered to be positive. It is, also, assumed that Buyer and Supplier known their own probabilistic market demands, which are considered to be continuous random variables. Let X and Z represent market demands, with their respective probability density functions, $f(\cdot)$ and $h(\cdot)$, and cumulative distribution functions, $F(\cdot)$ and $H(\cdot)$ that satisfy $F(0) = 0$ and $H(0) = 0$.

For each company, the medium-term production capacity cost is considered to be quadratic in order to reflect that some resources necessary for expanding the "soft" capacity (i.e. without doing investment of capital in fixed assets) as, for example, overtime and outsourcing labor force become more expensive than regular laboral force. The medium-term capacity cost-functions are defined by $\Psi_B(C) = a_B \cdot C^2$ and $\Psi_S(C) = a_S \cdot C^2$, respectively, for Buyer and Supplier.

4.1.1 The companies deciding independently

Under independent planning, Buyer and Supplier make their medium-term production-capacity decisions without considering any agreement, or contract, between them, so the markets are their sole trading opportunities. Thus, given the medium-term production-capacity, the expected operational profit for each company corresponds simply to the production margin (of selling to the market) times the quantity expected to be sold. Buyer's and Supplier's capacity problems are similar and defined, respectively, by Problem (1) and Problem (2) presented in Proposition 1. Also, each one of these problems has a unique optimal solution that satisfies the Equation (4-3) for the Buyer's case and Equation (4-4) for the Supplier's one, as presented in Proposition 2.

Proposition 1 Buyer's and Supplier's capacity problems are defined, respectively, by Problem (4-1) and Problem (4-2), whose objective functions are given, respectively, by the expected profits EP_B and EP_S.

$$\max_{0\le C}\left\{-\Psi_B(C)+E_X\Big[\Pi_{B|\varsigma}(C,x)\Big]\right\}=\max_{0\le C}\left\{-a_B\cdot C^2+\pi_B\left[C-\int_0^C F(x)dx\right]\right\} \tag{4-1}$$

$$\max_{0\le C}\left\{-\Psi_S(C)+E_{Y,Z}\Big[\Pi_{S|\varsigma}(C,z)\Big]\right\}=\max_{0\le C}\left\{-a_S\cdot C^2+\pi_S\left[C-\int_0^C H(z)dz\right]\right\} \tag{4-2}$$

Proposition 2 Buyer's and Supplier's capacity problems have a unique optimal solutions that are denoted, respectively, by C_B and C_S and satisfy, respectively, Equation (4-3) and Equation (4-4).

$$-2a_B\cdot C_B+\pi_B\Big[1-F(C_B)\Big]=0 \tag{4-3}$$

$$-2a_S\cdot C_S+\pi_S\Big[1-H(C_S)\Big]=0 \tag{4-4}$$

Since Buyer's capacity cost-function and expected operational profit-function under independent planning are given by, respectively, $E_X\Big[\Pi_{B|\varsigma}(C,x)\Big]=\pi_B\left[C-\int_0^C F(x)dx\right]$ and $-a_B\cdot C^2$, the marginal cost and expected profit associated to them are defined, respectively, by $\frac{d}{dC}\left(-a_B\cdot C_B^2\right)=-2a_B\cdot C_B$ and $\frac{d}{dC}\left(E_X\Big[\Pi_{B|\varsigma}(C,x)\Big]\right)=\pi_B\Big[1-F(C_B)\Big]$. Thus, the condition given by Equation (4-3) is simply the first-order optimality condition for the variable in his capacity problem, i.e. his optimal medium-term capacity decision correspond to the capacity-value which happens to equalize these marginal functions. Analogously for Supplier, the condition given by Equation (4-4) is simply the first-order optimality condition for the variable in her capacity problem, i.e. the intersection of the marginal capacity cost with the marginal expected operational profit.

According to Proposition 1, the Buyer-Supplier dyad's performance under *independent planning* (EP_{IP}) is given by Equation (4-5), where C_B and C_S represent the optimal capacity decisions, respectively, for Buyer and Supplier acting independently, that is, they satisfy the conditions given in Proposition 2.

$$\begin{aligned} EP_{IP} = &-\left(a_B \cdot C_B^2 + a_S \cdot C_S^2\right) + \\ &+ \pi_B \left[C_B - \int_0^{C_B} F(x)dx\right] + \pi_S \left[C_S - \int_0^{C_S} H(z)dz\right] \end{aligned} \tag{4-5}$$

4.1.2
The companies making decisions under the contract conditions

Under the contract situation, Buyer has the same information to decide the capacity decision under and without the contract, so Buyer's capacity problems are similar in both cases. However, Buyer's expected operational profit differs from the one under independent planning in the payment derived from the contract. Buyer's expected material (i.e. input) cost depends on the sources from which the material is bought, i.e. uniquely from the Supplier or from both sources (market and Supplier). Hence, Buyer's expected operational profit under the contract ζ, $E_X\left[\Pi_{B|\zeta}(C,x)\right]$, results to be defined by a function whose algebraic definition changes at the capacity commitment level, R, which is the threshold to the amount to be bought from Supplier. However, that function is continuous and concave, and consequently, Buyer's capacity problem has a unique optimal solution. These results are presented in Proposition 3 and Proposition 4.

Proposition 3 Buyer's capacity problem under the contract $\zeta = (R, d, t)$ is defined by Problem (4-6), whose objective function is continuous.

$$\max_{0 \leq C} \{EP_{B|\zeta}\} = \max_{0 \leq C} \left\{ -a_B \cdot C^2 + E_X \left[\Pi_{B|\zeta}(C, x)\right] \right\} \tag{4-6}$$

where

$$E_X \left[\Pi_{B|\zeta}(C, x)\right] = \pi_B \left[C - \int_0^C F(x)dx\right] + \begin{cases} I_1(C) & , C \leq R \\ I_2 & , C > R \end{cases}$$

with $I_1(C) = (d \cdot p_m + t) \left[C - \int_0^C F(x)dx\right] - t \cdot R$, and $I_2 = (d \cdot p_m + t) \left[R - \int_0^R F(x)dx\right] - t \cdot R$.

Proposition 4 Buyer's capacity problem under the contract $\zeta = (R, d, t)$ has a unique optimal solution, here denoted by $C_{B|\zeta}$ and is given (implicitly) by Equation (4-7).

$$C_{B|\zeta} = \begin{cases} C_1^* & , \text{ if } EP_{B|\zeta}(C_1^*) \geq EP_{B|\zeta}(C_2^*) \\ C_2^* & , \text{ if } EP_{B|\zeta}(C_1^*) < EP_{B|\zeta}(C_2^*) \end{cases} \tag{4-7}$$

where

$$C_1^* = \arg \max_{C \in [0,R]} \left\{EP_{B|\zeta}(C)\right\} = \begin{cases} C_1 & , \text{ if } C_1 \in [0, R) \\ R & , \text{ if } C_1 \in [R, \infty) \end{cases}$$

$$C_2^* = \arg \max_{C \in [R,\infty)} \left\{EP_{B|\zeta}(C)\right\} = \begin{cases} C_2 & , \text{ if } C_2 \in (R, \infty) \\ R & , \text{ if } C_2 \in [0, R] \end{cases}$$

with $EP_{B|\zeta}(C) = -a_B \cdot C^2 + E_X\left[\Pi_{B|\zeta}(C, x)\right]$, and C_1 and C_2 satisfying, respectively, $-2a_B \cdot C_1 + (\pi_B + d \cdot p_m + t) \left[1 - F(C_1)\right] = 0$ and $-2a_B \cdot C_2 + \pi_B\left[1 - F(C_2)\right] = 0$.

Note that Buyer's expected operational profit-function under the contract situation differs of the one under independent planning in the term $I_1(C)$ if $C \leq R$, or I_2 if $C > R$. The expressions for $I_1(C)$ and I_2 given in Proposition 3 can be interpreted as Buyer paying the penalty for the entire reserved capacity $-t \cdot R$ and receiving the discount, as well as recovering the penalty, for the units that he orders, i.e. $(d \cdot p_m + t) \left[C - \int_0^C F(x)dx\right]$ if $C \leq R$, or $(d \cdot p_m + t) \left[R - \int_0^R F(x)dx\right]$ if $C > R$.

The objective function for Buyer's capacity problem under the contract $\zeta = (R, d, t)$ is continuous and defined by parts. So, Buyer's global-optimal capacity decision is taken among the local-optimal capacities, according to the Equation (4-7). Those local-optimal capacities are denoted by C_1^* and C_2^* in

Proposition 4, and they correspond to the capacity-values that satisfy the first-order optimality condition (C_1 and C_2), or to the committed capacity level (R). Since the marginal functions associated to the capacity cost-function and expected operational profit-function are defined, respectively, by $\frac{d}{dC}\left(-a_B \cdot C^2\right) = -2a_B \cdot C$ and $\frac{d}{dC}\left(E_X\left[\Pi_{B|\zeta}(C,x)\right]\right) = (\pi_B + d \cdot p_m + t)\left[1 - F(C_1)\right]$ if $C \leq R$, or $\frac{d}{dC}\left(E_X\left[\Pi_{B|\zeta}(C,x)\right]\right) = \pi_B\left[1 - F(C_2)\right]$ if $C > R$, the capacity-values for C_1 and C_2 must satisfy, respectively, $-2a_B \cdot C_1 + (\pi_B + d \cdot p_m + t)\left[1 - F(C_1)\right] = 0$ and $-2a_B \cdot C_2 + \pi_B\left[1 - F(C_2)\right] = 0$.

Supplier's expected revenue, under the contract situation, comprises the ones from selling the material to Buyer and to the market. Let a continuous random variable Y represent the material order from Buyer to Supplier, and $g(\cdot)$ and $G(\cdot)$, respectively, her mixed probability-density function and cumulative distribution function. Since Buyer's production is bounded by his medium-term production capacity, his material entire requirement is limited to that capacity. Therefore, the mixed probability-density function $g(\cdot)$ has always limited range, whose uppermost value (y_{max}) corresponds to given Buyer's medium-term capacity decision, that is, $y_{max} = C_{B|\zeta}$. That function is defined by the expression in Definition (4-8).

$$g(y) = \begin{cases} f(y) & , \ y < y_{max} = C_{B|\zeta} \\ 1 - F(y_{max}) & , \ y = y_{max} = C_{B|\zeta} \end{cases} \tag{4-8}$$

Despite of depending on Buyer's material order distribution, Supplier's expected operational profit ($E_{Y,Z}\left[\Pi_{S|\zeta}(C,y,z)\right]$) has unique algebraic expression of definition along its domain, which depends on the relation between the values of y_{max} and R, the capacity commitment level specified in the contract. Also, due to the forced compliance assumption, Supplier's capacity has lower bound equal to R. The formulation for that problem is presented in Proposition 5, while the unique optimal solution is given, in an implicit way, by Proposition 6.

Proposition 5 Given the contract $\zeta = (R, d, t)$, Supplier's capacity problem is defined by Problem (4-9), whose objective function is continuous.

$$\max_{R \leq C} \{EP_{S|\zeta}\} = \max_{R \leq C} \left\{ -a_S \cdot C^2 + E_{Y,Z}\left[\Pi_{S|\zeta}(C, y, z)\right] \right\} \tag{4-9}$$

where $E_{Y,Z}\left[\Pi_{S|\zeta}(C, y, z)\right]$ is given by Equation(4-10), with $K = y_{max}$ if $y_{max} \leq R$, or $K = R$ if $y_{max} > R$.

$$E_{Y,Z}\left[\Pi_{S|\zeta}(C, y, z)\right] = \pi_S \left[C - \int_0^K D_x(C) f(x) dx - D_K(C)\ (1 - F(K)) \right] + J_K \tag{4-10}$$

with $D_x(C) = \int_0^{C-x} H(z)dz$, $D_K(C) = \int_0^{C-K} H(z)dz$, $J_{y_{max}} = -(d \cdot p_m + t)\ E[Y] + t \cdot R$, and $J_R = -(d \cdot p_m + t) \left[R - \int_0^R F(x)dx \right] + t \cdot R$.

Proposition 6 Supplier's capacity problem under the contract $\zeta = (R, d, t)$ has a unique optimal solution that is denoted by $C_{S|\zeta}$ and given (implicitly) by Equation (4-11).

$$C_{S|\zeta} = \begin{cases} C^\circ & , \text{ if } C^\circ \in (R, \infty) \\ R & , \text{ if } C^\circ \in [0, R] \end{cases} \tag{4-11}$$

where C° is the unique stationary point for the expression that defines Supplier's expected profit, which satisfies the next Equation (4-12) with $K = y_{max}$ if $y_{max} \leq R$, or $K = R$ if $y_{max} > R$.

$$-2a_S \cdot C^\circ + \pi_S \left[1 - \int_0^K H(C^\circ - x) f(x) dx - H(C^\circ - K)\Big(1 - F(K)\Big) \right] = 0 \tag{4-12}$$

The Supplier's expected operational profit-function under contract has, in contrast with Buyer's one, a definition along its domain, which is given by Equation(4-10), with $K = y_{max}$ if $y_{max} \leq R$, or $K = R$ if $y_{max} > R$. That expression is more involved than the one under independent planning, because the quantity expected to be sold depends on the market demand and on Buyer's material order, which are considered to be independent. The expressions for $J_{y_{max}}$ and J_R given in Proposition 5 can be interpreted as Supplier receiving the penalty for the entire reserved capacity $t \cdot R$ and giving the discount, as well as reembursing the penalty, for the units ordered by Buyer, i.e. $-(d \cdot p_m + t)\ E[Y]$ if $y_{max} \leq R$, or $-(d \cdot p_m + t) \left[R - \int_0^R F(x)dx \right]$ if $y_{max} > R$. This makes clear the equivalence of the contract proposed with the DR contract.

Again, Supplier's optimal capacity decision given by Equation (4-11) corresponds to the capacity-value that satisfies the first-order optimality condition

(C°), or the committed capacity level (R) because the Supplier's capacity has a lower bound equal to that level. Since the marginal functions associated to Supplier's capacity cost-function and expected operational profit-function are defined, respectively, by $\frac{d}{dC}\left(-a_S \cdot C^2\right) = -2a_S \cdot C$ and $\frac{d}{dC} E_{Y,Z}\left[\Pi_{S|\zeta}(C,y,z)\right] = \pi_S\left[1 - \int_0^K H(C^\circ - x) f(x)dx - H(C^\circ - K)\Big(1 - F(K)\Big)\right]$, the capacity-value for C° must satisfy Equation (4-12), where $K = y_{max}$ if $y_{max} \leq R$, or $K = R$ if $y_{max} > R$. That last equation is simply the capacity value at the intersection of the capacity cost-function with the expected operational profit-function.

Note that, in getting the assessment for the dyad's expected profit on the optimal capacity decisions, the terms involved in the expressions of the companies' expected profits that are associated to the monetary transference generated by the contract ζ vanish. Since that $C_{B|\zeta} = y_{max}$, those terms are $I_1(C)$ and J when $C_{B|\zeta} = y_{max} \leq R$, while the ones are I_2 and K when $C_{B|\zeta} = y_{max} > R$. Thus, the companies' joint expected profit or, equivalently, the Buyer-Supplier dyad's expected profit under the contract conditions $(EP_{D|\zeta})$ is given by the Equation (4-13), in which $C_{B|\zeta}$ and $C_{S|\zeta}$ represent to the optimal capacity decisions, respectively, for Buyer and Supplier taking into account the conditions of the contract ζ.

$$EP_{D|\zeta} = -\Big(a_B \cdot C_{B|\zeta}^2 + a_S \cdot C_{S|\zeta}^2\Big) + \pi_B\left[C_{B|\zeta} - \int_0^{C_{B|\zeta}} F(x)dx\right] + \\ + \pi_S\left[C_{S|\zeta} - \int_0^K D_x(C_{S|\zeta}) f(x)dx - D_K(C_{S|\zeta})\ (1 - F(K))\right] \tag{4-13}$$

where $K = C_{B|\zeta} = y_{max}$ if $C_{B|\zeta} = y_{max} \leq R$, or $K = R$ if $C_{B|\zeta} = y_{max} > R$, and $D_{y_{max}}(\cdot)$ and $D_R(\cdot)$ are as defined in Proposition 5.

4.1.3
The companies being coordinated by central planning

If the companies are coordinated by central planning, that is, they can be considered as producing units of a single system. The medium-term production capacities are decided simultaneously and obtained by the, here called, centralized system's capacity problem, whose objective function is in terms of the Buyer's and Supplier's capacities, respectively, C_B and C_S. The centralized system's expected profit $(EP_{CP}(C_B, C_S))$ corresponds to the companies' expected operational profit under central planning $(E_{X,Z}\left[\Pi_{CP}\right](C_B, C_S, x, z))$ minus the sum of the capacity cost of each producing company unit, or Buyer and Supplier. According to Proposition 7 and Proposition 8, the centralized system's problem is defined by Problem (14), whose unique optimal solution is given implicitly by Equation (4-16).

Proposition 7 The centralized system's capacity problem is equivalent to Problem (4-14), whose objective function correspond to $EP_{CP}(C_B, C_S)$ that is a strictly concave and continuous function.

$$\max_{(0,0)\leq(C_B,C_S)} \left\{-(a_B \cdot C_B^2 + a_S \cdot C_S^2) + E_{X,Z}\left[\Pi_{CP}(C_B, C_S, x, z)\right]\right\} \tag{4-14}$$

where

$$\begin{aligned} \mathrm{E}_{X,Z}\left[\Pi_{CP}(C_B, C_S, x, z)\right. &= \pi_B \left[C_B - \int_0^{C_B} F(x)dx\right] + \\ &+ \ \pi_S \begin{cases} \left[C_S - \int_0^{C_S} D_x(C_S)f(x)dx\right] & , \ C_S \leq C_B \\ \left[C_S - \int_0^{C_B} D_x(C_S)f(x)dx - \right. \\ \quad \left. -\Big(1 - F(C_B)\Big)\int_0^{C_S - C_B} H(z)dz\right] & , \ C_S > C_B \end{cases} \end{aligned} \tag{4-15}$$

with $D_x(C) = \int_0^{C-x} H(z)dz$, $F(\cdot)$ and $H(\cdot)$ are, respectively, the cumulative distributions for the market demands for the product and material (X and Z).

Proposition 8 The centralized system's capacity problem admits a unique optimal solution, which is denoted by (C_B^*, C_S^*) and is given (implicitly) by Equation (4-16).

$$(C_B^*, C_S^*) = \begin{cases} (C_B^{a*}, C_S^{a*}) & , \text{ if } C_S^{a*} > C_B^{a*}, \ C_S^{b*} \leq C_B^{b*} \\ & \quad \text{and } EP_{CP}(C_B^{a*}, C_S^{a*}) \geq EP_{CP}(C_B^{b*}, C_S^{b*}), \\ & \quad \text{or if } C_S^{a*} > C_B^{a*}, \ C_S^{b*} > C_B^{b*} \\ (C_B^{b*}, C_S^{b*}) & , \text{ if } C_S^{a*} > C_B^{a*}, \ C_S^{b*} \leq C_B^{b*} \\ & \quad \text{and } EP_{CP}(C_B^{a*}, C_S^{a*}) < EP_{CP}(C_B^{b*}, C_S^{b*}), \\ & \quad \text{or if } C_S^{a*} \leq C_B^{a*}, \ C_S^{b*} \leq C_B^{b*} \\ (C_B^{*}, C_B^{*}) & , \text{ if } C_S^{a*} \leq C_B^{a*} \text{ and } C_S^{b*} > C_B^{b*} \end{cases} \tag{4-16}$$

where (C_B^{a*}, C_S^{a*}) and (C_B^{b*}, C_S^{b*}) are the stationary points of the expected profit in each region, $C_S > C_B$ and $C_S \leq C_B$, while C_B^* is the value that maximizes the boundary line between these regions, and $EP_{CP}(C_B, C_S) = -(a_B \cdot C_B^2 + a_S \cdot C_S^2) + E_{X,Z}\left[\Pi_{CP}(C_B, C_S, x, z)\right]$. These points satisfy the following conditions:

$$\begin{cases} (1-F(C_B^{a*}))H(C_S^{a*}-C_B^{a*}) = \frac{2a_B \cdot C_B^{a*} - \pi_B(1-F(C_B^{a*}))}{\pi_S} \\ \pi_S \int_0^{C_B^{a*}} H(C_S^{a*}-x)f(x)dx = -2a_B \cdot C_B^{a*} - 2a_S \cdot C_S^{a*} + \pi_B(1-F(C_B^{a*})) + \pi_S \end{cases} \tag{4-17a}$$

$$\begin{cases} F(\cdot) \cap L_B(\cdot) = \{C_B^{b*}\} \\ A_x(\cdot) \cap L_S(\cdot) = \{C_S^{b*}\} \end{cases} \tag{4-17b}$$

$$C_B^* = \arg\max_{0 \le C_B} \Big\{ -(a_B + a_S) \cdot C_B^2 + E_{X,Z}\Big[\Pi_{CP}(C_B, C_B, x, z)\Big]\Big\} \tag{4-17c}$$

with $L_k(C) = 1 - \frac{2a_k}{\pi_k} \cdot C$, $k \in \{B, S\}$, $A_x(C) = \int_0^C H(C-x)f(x)dx$, and $F(\cdot)$ and $H(\cdot)$ are, respectively, the cumulative distributions for the market demands for X and Z.

The centralized system's expected operational profit-function happens to be a continuous function defined by parts according to the relation between the manufacturing units' capacity-values, C_B and C_S. So, the globally optimal capacity decision (C_B^*, C_S^*) is taken among the locally optimal capacity decisions, according to the Equation (4-16). These locally optimal decisions correspond to the stationary point in each part of the expected operational-profit definition, (C_B^{a*}, C_S^{a*}) or (C_B^{b*}, C_S^{b*}), or the boundary point between the definition parts, (C_B^*, C_B^*). Arranging the expressions that define the first-order optimality condition, the stationary points are given, respectively, by the equations (4-17a) and (4-17b), while the expression that define the first-order optimality condition to determine the boundary point is given by the equation (4-17c).

The Buyer-Supplier dyad's performance under *central planning*, EP_{CP}^*, is given by the centralized system's maximum expected profit , that is $EP_{CP}^* = EP_{CP}(C_B^*, C_S^*)$, where (C_B^*, C_S^*) is defined by Equation(4-16).

4.2 Appraisal of the companies' optimal capacity decisions

It is possible to establish a relation between the companies' optimal capacity decisions under a generic contract and the ones under independent planning. In general, the companies' optimal capacities under a generic contract are greater or equal to the ones under independent planning. It is characteristic of elimination of double marginalization. In particular, that relation is strict for the Supplier's optimal decisions when they are not equal to the upper bound. There are upper bounds for the capacity decisions under both situations, under and without of the contract. These results are presented in Proposition 9 and Proposition 10, respectively, for Buyer and Supplier.

Proposition 9 Buyer's optimal capacity under the contract situation is greater than or equal to his capacity decision under independent planning, and these capacities are upperly bounded by $\frac{\pi_B}{2a_B}$. Also, the optimal solution for Buyer's capacity problem $C_{B|\zeta}$ can be expressed by Equation (4-18), where C_B is Buyer's optimal capacity under independent planning.

$$C_{B|\zeta} = \begin{cases} C_1 & \text{, if } C_1 \in [0, R) \\ C_B & \text{, if } C_B \in \left(R, \frac{\pi_B}{2a_B}\right] \\ R & \text{, if } C_1 \notin [0, R) \wedge C_B \notin \left(R, \frac{\pi_B}{2a_B}\right] \end{cases} \tag{4-18}$$

where C_1 and C_B satisfy, respectively, $-2a_B \cdot C_1 + (\pi_B + d \cdot p_m + t)\Big[1 - F(C_1)\Big] = 0$ and $-2a_B \cdot C_B + \pi_B\Big[1 - F(C_B)\Big] = 0$.

Proposition 10 Supplier' optimal capacity under the contract situation is always larger than her capacity decision under independent planning, unless these capacities are equal to the bounding value given by $\frac{\pi_S}{2a_S}$.

Note that the relation between the optimal capacities does not imply that a given contract is viable for both companies. In fact, Buyer will be better off under the contract only if the cost of increasing his capacity is less than the increase of the expected operational profit derived from the increased capacity. From the Supplier's point of view, the forced compliance assumption forces her to increase her capacity, in relation to the one under independent planning, if her capacity decision under the contract conditions is less than the capacity commitment level. So, for adhering to the contract conditions, Supplier must set her capacity at the level committed and, consequently, will be worse off under the contract.

4.3 The contract problem and optimization procedure

In the previous sections the optimal capacities have been characterized under a given contract, under independent planning, as well as under central planning. It is unfortunate that the optimal profit conditions and profits are not explicit functions of the contract parameters, because this precludes the analytical assessment of optimal contracts. Hence most of the analysis must be made through particular numerical instances.

The contract optimization procedure intends to determine optimal contracts in the continuous parameter space. That is, sets of parameters that define contracts that maximize the companies' expected joint profit must be found by searching a superset of the optimal parameter space. As it was not possible to get the closed expressions for the partial derivates for the companies' expected profits, a derivative free pattern search algorithm will be defined to find some locally optimal contracts. Obviously, only viable contracts are of interest. However, the viability constraints involve non-linear functions without explicit expressions for their partial derivates, which makes it quite complex to use a constrained search algorithm.

Clearly, ignoring the viability constraints, the contracts that optimize to the dyad's expected profit can be unviable for some, or both, companies. Thus, a practical stance was taken here. The search will be made in a subset of the continuous parameter space Ω defined by some bounds that exclude uninteresting contracts. Optimal contracts in this reduced subset will be searched, starting from different points. The contracts that are viable for both companies will be retained at each iteration of the optimization search process, and the best of them will be considered as viable and (local) optimal contract, which will define the set denote by Ω_{VOC}. To obtain a large variety of optimal contracts a grid of starting points will be used, which defines a discrete set of initial contracts that is denoted by $\Omega_\circ$.

Note that, considering Proposition 10, it does not make sense to consider the capacity commitment level R larger than the Supplier's maximum economic capacity. Thus, the continuous parameter space Ω is defined by Set (4-19), while an upper bound for the penalty parameter will be considered to define the discrete set of initial contracts $\Omega_\circ$.

$$\Omega = \left\{ (R, d, t) : R \in \left[0, \frac{\pi_S}{2a_S} \right], d \in [0,1], t \geq 0 \right\} \tag{4-19}$$

Therefore, the *contract problem* based on the dyad's expected profit $EP_{D|\zeta}$ (where $EP_{D|\zeta} = EP_{B|\zeta} + EP_{S|\zeta}$) is defined by Problem (4-20), whose objective function is given by the sum of the companies' expected operational profit minus the sum of the companies' capacity cost.

$$\max_{\zeta\in\Omega}\left\{-\left(a_B\cdot C^2_{B|\zeta}+a_S\cdot C^2_{S|\zeta}\right)+E_X\left[\Pi_{B|\zeta}(C_{B|\zeta},x)\right]+E_{Y,Z}\left[\Pi_{S|\zeta}(C_{S|\zeta},y,z)\right]\right\} \tag{4-20}$$

where $C_{B|\zeta}$ and $C_{S|\zeta}$ are the optimal capacity decisions under the contract ζ, respectively, for Buyer and Supplier.

The pattern search algorithm used is based on the one developed by Lewis and Torczon (1999), which is an algorithm defined for bound-constrained maximization that considers the possibility of unbounded values for the variables. Specifically, at each iteration, the pattern search algorithm evaluates and compares the Buyer-Supplier dyad's expected profit under a trial-move contract and the current contract of the iteration. The trial contract is the best selected among the feasible contracts (i.e. within the given boundaries) obtained by exploratory moves defined in the pattern search. In this work, the bound-constrained exploratory moves are defined in the direction of the axes and the principal diagonal of the quadrants defined by the current contract as origin. The search algorithm is stopped when the gain of the dyad's expected profit obtained by the last iteration is less than a given value.