2
Linear state space models and the Kalman filtering

2.1
The model

A *linear wide sense* state space model for an observable p-variate stochastic process Y_t, defined on an appropriate probability space $(\Omega, \mathcal{F}, \mathcal{P})$, is described by the following set of equations:

$$\begin{aligned} Y_t &= Z_t\alpha_t + d_t + \varepsilon_t \\ \alpha_{t+1} &= T_t\alpha_t + c_t + R_t\eta_t \\ \alpha_1 &\sim (a_1, P_1). \end{aligned} \tag{2-1}$$

The first equation is usually called the *measurement equation* and the second is known as the *state equation*. The unobservable m-variate process α_t is termed *state*. The error terms ε_t and η_t are respectively p-variate and r-variate second-order processes that are uncorrelated in time and from each other, with $var(\varepsilon_t) = H_t$ and $var(\eta_t) = Q_t$. The remaining *system matrices* $Z_t, d_t, H_t, T_t, c_t, R_t$ and Q_t evolve deterministically.

2.2
The Kalman equations

In this Thesis, I will adopt the following notation:

- $a_{t|j}$ is a (an equivalence class of) random vector(s) with coordinates $a_{ti|j}$, $i = 1, \ldots, m$, representing the unique linear orthogonal projection (cf. Kubrusly, 2001, Theorem 5.52), evaluated on each (equivalence class of) coordinate(s) α_{ti} of α_t, onto $S' \equiv span\{1, Y_{11}, \ldots, Y_{1p}, \ldots, Y_{j1}, \ldots, Y_{jp}\} \subseteq L_2 \equiv L_2(\Omega, \mathcal{F}, \mathcal{P})$ - the subjacent topology is that induced by the usual inner product, which is given by $< X, Y >\equiv E(XY) = \int_\Omega X(\omega)Y(\omega)\mathcal{P}(d\omega), \forall X, Y \in L_2$.
- $P_{t|j} \equiv E\left[(\alpha_t - a_{t|j})(\alpha_t - a_{t|j})'\right]$;
- $\upsilon_t \equiv Y_t - Z_t a_{t|t-1} - d_t$ and $F_t \equiv E(\upsilon_t\upsilon_t') = Z_t P_{t|t-1} Z_t' + H_t$.

The *Kalman filtering* (prediction, updating and smoothing) gives the above orthogonal projections evaluations and the corresponding mean square error matrices. The corresponding equations are given as follows:

– Prediction equations

$$\begin{aligned} a_{t+1|t} &= T_t a_{t|t} + c_t \\ P_{t+1|t} &= T_t P_{t|t} T_t' + R_t Q_t R_t' \end{aligned} \tag{2-2}$$

– Updating or filtering equations

$$\begin{aligned} a_{t|t} &= a_{t|t-1} + P_{t|t-1} Z_t' F_t^{-1} \upsilon_t \\ P_{t|t} &= P_{t|t-1} - P_{t|t-1} Z_t' F_t^{-1} Z_t P_{t|t-1} \end{aligned} \tag{2-3}$$

– Smoothing equations (for a given $n \geq t$)

$$\begin{aligned} a_{t|n} &= a_{t|t-1} + P_{t|t-1} r_{t-1} \\ r_{t-1} &= Z_t' F_t^{-1} \upsilon_t + (T_t - T_t P_{t|t-1} Z_t' F_t^{-1} Z_t)' r_t \\ P_{t|n} &= P_{t|t-1} - P_{t|t-1} N_{t-1} P_{t|t-1} \\ N_{t-1} &= Z_t' F_t^{-1} Z_t + (T_t - T_t P_{t|t-1} Z_t' F_t^{-1} Z_t)' N_t (T_t - T_t P_{t|t-1} Z_t' F_t^{-1} Z_t) \\ r_n &= 0 \; and \; N_n = 0 \end{aligned} \tag{2-4}$$

Details concerning the derivations of these formulae are found in Harvey (1989), Brockwell and Davis (1991), Harvey (1993), de Jong (1989), Hamilton (1994), Tanizaki (1996), Durbin and Koopman (2001), Brockwell and Davis (2003) and Shumway and Stoffer (2006).

2.3 Introducing linear restrictions

From now on, it is assumed that the process α_t in (2-1) satisfies linear restrictions as follows:

Assumption 1 *The random vectors α_t satisfy the following (possibly time varying) linear restrictions*

$$A_t \alpha_t = q_t, \tag{2-5}$$

where, for each t, A_t is a $k \times m$ matrix and q_t is a $k \times 1$ (possibly random) vector.

Observe that the restrictions enunciated in eq.(2-5) are rather general. In fact, it encapsulates *affine* restrictions of the kind $A_t \alpha_t + b_t = q_t$ by defining $q_t' = q_t - b_t$ and allows the number of restrictions k to be time-varying. In practical situations, justification of such constraints in (2-5) arises naturally from the characteristics of the problem being modeled; see for instance the restrictions imposed on a demand system problem in Doran and Rambaldi (1997).

In the remaining of the Thesis, Assumption 1 will be considered in almost every topic to be discussed and, in due course, it may be added with some further structure on the linear restrictions.