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## Apêndice A

### Termos das equações diferenciais de movimento para pórticos

Neste apêndice são apresentados os termos das equações de movimento para pórticos, obtidos na seção 2.3. Os termos  $T_{ij}$ , usados na forma dimensional das equações, são apresentados na seção A.1 enquanto que os termos  $T_{ij}^*$  são apresentados por sua vez na seção A.2. Em ambos os termos, o índice  $i$  faz referência à equação, assim  $i=1$  se refere à eq. (2-51) e  $i=2$  se refere à eq. (2-52). Enquanto  $j$  se refere à ordem da não-linearidade dos termos, assim  $j=0$  se refere aos termos constantes,  $j=1$  aos termos lineares,  $j=2$  aos termos com não-linearidade quadrática e assim por diante.

#### A.1

##### Termos da equação na forma dimensional

$$T_{1,0} = 0 ;$$

$$T_{1,1} = \rho A u_{,tt} + c u_{,t} + (P - EA) u_{,xx} ;$$

$$T_{1,2} = -EI (w_{,x} w_{,xxx} + w_{,xx} w_{,xxx}) - EA (w_{,x} w_{,xx} + 3u_{,x} u_{,xx}) ;$$

$$T_{1,3} = EI (5w_{,x} u_{,xx} w_{,xxx} + w_{,x}^2 u_{,xxx} + 4w_{,x} w_{,xx} u_{,xxx} + 2w_{,xx}^2 u_{,xx} + 3w_{,x} u_{,x} w_{,xxx} + 3u_{,x} w_{,xx} w_{,xxx}) - \frac{1}{2} EA (w_{,x}^2 u_{,xx} + 3u_{,x}^2 u_{,xx} + 2w_{,x} u_{,x} w_{,xx}) ;$$

$$T_{1,4} = 2EI (2w_{,x}^3 w_{,xxx} - 4w_{,x}^2 u_{,xx} u_{,xxx} - 6w_{,x} w_{,xx} u_{,xx}^2 - 8w_{,x} u_{,x} w_{,xx} u_{,xxx} - 3u_{,x}^2 w_{,xx} w_{,xxx} - 2w_{,x}^2 u_{,xx} u_{,xxx} + 6w_{,x} w_{,xx}^3 - 15u_{,x}^2 w_{,xx} w_{,xxx} - 16u_{,x} w_{,xx}^2 u_{,xx} + 9w_{,x}^2 w_{,xx} w_{,xxx} - 10w_{,x} u_{,x} u_{,xx} w_{,xxx}) ;$$

$$T_{1,5} = 2EI (-6w_{,x} u_{,x} w_{,xx}^3 - 5w_{,x}^3 u_{,xx} w_{,xxx} - w_{,x}^4 u_{,xxx} - 8w_{,x}^3 w_{,xx} u_{,xxx})$$

$$\begin{aligned}
& + 28u_{,x}^2 w_{,xx}^2 u_{,xx} - 15w_{,x}^2 w_{,xx}^2 u_{,xx} + 20w_{,x} u_{,x}^2 w_{,xx} u_{,xxx} + w_{,x} u_{,x}^3 w_{,xxxx} \\
& + 13w_{,x} u_{,x}^2 u_{,xx} w_{,xxx} + 5w_{,x}^2 u_{,x}^2 u_{,xxx} + 5w_{,x}^2 u_{,xx}^3 - w_{,x}^3 u_{,x} w_{,xxxx} \\
& - 9w_{,x}^2 u_{,x} w_{,xx} w_{,xxx} + 30w_{,x} u_{,x} w_{,xx} u_{,xx}^2 + 13u_{,x}^3 w_{,xx} w_{,xxx} + 20w_{,x}^2 u_{,x} u_{,xx} u_{,xxx} );
\end{aligned}$$

$$\begin{aligned}
T_{1,6} = EI & \left( 32w_{,x}^3 u_{,x} w_{,xx} u_{,xxx} + 24w_{,x}^3 w_{,xx} u_{,xx}^2 + 36w_{,x} u_{,x}^2 w_{,xx}^3 \right. \\
& - 108w_{,x} u_{,x}^2 w_{,xx} u_{,xx}^2 - 17w_{,x}^4 w_{,xx} w_{,xxx} - 72u_{,x}^3 w_{,xx}^2 u_{,xx} - w_{,x}^5 w_{,xxxx} - 48w_{,x} u_{,x}^3 w_{,xx} u_{,xxx} \\
& + 3w_{,x} u_{,x}^4 w_{,xxxx} + 8w_{,x}^4 u_{,xx} u_{,xxx} - 36w_{,x}^2 u_{,x} u_{,xx}^3 - 21u_{,x}^4 w_{,xx} w_{,xxx} \\
& + 4w_{,x}^4 u_{,x} u_{,xxxx} - 12w_{,x}^2 u_{,x}^3 u_{,xxxx} - 24w_{,x}^3 w_{,xx}^3 + 20w_{,x}^3 u_{,x} u_{,xx} w_{,xxx} - 12w_{,x} u_{,x}^3 u_{,xx} w_{,xxx} \\
& \left. + 84w_{,x}^2 u_{,x} w_{,xx}^2 u_{,xx} + 2w_{,x}^3 u_{,x}^2 w_{,xxxx} + 42w_{,x}^2 u_{,x}^2 w_{,xx} w_{,xxx} - 72w_{,x}^2 u_{,x}^2 u_{,xx} u_{,xxx} \right);
\end{aligned}$$

$$\begin{aligned}
T_{1,7} = EI & \left( -w_{,x}^5 u_{,x} w_{,xxxx} - 81u_{,x}^5 w_{,xx} w_{,xxx} - 162u_{,x}^4 w_{,xx}^2 u_{,xx} + 5w_{,x}^5 u_{,xx} w_{,xxx} \right. \\
& + 12w_{,x}^5 w_{,xx} u_{,xxx} + 24w_{,x}^4 u_{,xx} w_{,xx}^2 + 9w_{,x}^2 u_{,x}^4 u_{,xxxx} - 6w_{,x}^4 u_{,x}^2 u_{,xxxx} - 9w_{,x} u_{,x}^5 w_{,xxxx} \\
& + 6w_{,x}^3 u_{,x}^3 w_{,xxxx} + 54w_{,x}^2 u_{,x}^2 u_{,xx}^3 + 60w_{,x} w_{,xx}^3 u_{,x}^3 - 24w_{,x}^3 u_{,x} w_{,xx}^3 \\
& - 48w_{,x}^3 u_{,x}^2 w_{,xx} u_{,xxx} + 36w_{,x} u_{,x}^4 w_{,xx} u_{,xxx} + 78w_{,x}^2 u_{,x}^3 w_{,xx} w_{,xxx} - 17w_{,x}^4 u_{,x} w_{,xx} w_{,xxx} \\
& 18w_{,x}^2 u_{,x}^2 w_{,xx}^2 u_{,xx} + 36w_{,x} u_{,x}^4 w_{,xx} u_{,xxx} - 72w_{,x}^3 u_{,x} w_{,xx} u_{,xx}^2 + 108w_{,x} u_{,x}^3 w_{,xx} u_{,xx}^2 \\
& - 24w_{,x}^4 u_{,x} u_{,xx} u_{,xxx} + 72w_{,x}^2 u_{,x}^3 u_{,xx} u_{,xxx} - 6w_{,x}^3 u_{,x}^2 u_{,xx} w_{,xxx} - 27w_{,x} u_{,x}^4 u_{,xx} w_{,xxx} \\
& \left. - 6w_{,x}^4 u_{,xx}^3 + w_{,x}^6 u_{,xxxx} \right);
\end{aligned}$$

$$T_{2,0} = X_0 \cos(wt);$$

$$T_{2,1} = \rho A w_{,tt} + c u_{,t} + P w_{,xx} - EI w_{,xxxx};$$

$$T_{2,2} = -EI (w_{,x} u_{,xxxx} + 4u_{,xx} w_{,xxx} + 3w_{,xx} u_{,xxx} + 2u_{,x} w_{,xxxx}) - EA (w_{,x} u_{,xx} + u_{,x} w_{,xx});$$

$$\begin{aligned}
T_{2,3} = EI & \left( 7w_{,x} u_{,xx} u_{,xxx} - 2w_{,x}^2 w_{,xxxx} - 8w_{,x} w_{,xx} w_{,xxx} + 8w_{,xx} u_{,xx}^2 \right. \\
& \left. + 3w_{,x} u_{,x} u_{,xxxx} + 9u_{,xx} w_{,xx} u_{,xxx} + 12u_{,x} u_{,xx} w_{,xxx} + 3u_{,x}^2 w_{,xxxx} - 2w_{,xx}^3 \right)
\end{aligned}$$

$$-\frac{1}{2}EA(u_{,x}^2 w_{,xx} + 3w_{,x}^2 w_{,xx} + 2w_{,x} u_{,x} u_{,xx});$$

$$T_{2,4} = 2EI \left( w_{,x}^3 u_{,xxxx} + 3u_{,x}^2 w_{,xx} u_{,xxx} + 8u_{,x} w_{,xx} u_{,xx}^2 - 14w_{,x} u_{,x} u_{,xx} u_{,xxx} \right. \\ \left. - 3u_{,x}^2 w_{,x} u_{,xxxx} + 3w_{,x}^2 w_{,xx} u_{,xxx} - 4w_{,x} u_{,xx}^3 + 12u_{,x}^2 u_{,xx} w_{,xxx} \right. \\ \left. + 2u_{,x}^3 w_{,xxxx} \right);$$

$$T_{2,5} = EI \left( -8w_{,x} u_{,x} u_{,xx}^3 - +8w_{,x}^3 w_{,xx} w_{,xxx} + w_{,x}^4 w_{,xxxx} + 2w_{,x}^3 u_{,xx} u_{,xxx} \right. \\ \left. + 6w_{,x}^2 u_{,xx}^2 w_{,xx} - 64u_{,x}^2 w_{,xx} u_{,xx}^2 - 2w_{,x} u_{,xx}^2 u_{,xx} u_{,xxx} - 2w_{,x}^3 u_{,x} u_{,xxxx} \right. \\ \left. - 6w_{,x}^2 u_{,x} w_{,xx} u_{,xxx} + 2w_{,x} u_{,xx}^3 u_{,xxxx} + 6w_{,x}^2 w_{,xx}^3 - 18u_{,x}^3 w_{,xx} u_{,xxx} \right. \\ \left. - 5u_{,x}^4 w_{,xxxx} - 40u_{,x}^3 u_{,xx} w_{,xxx} \right);$$

$$T_{2,6} = EI \left( -32 w_{,x} u_{,x}^3 w_{,xx} w_{,xxx} + 12 w_{,x}^3 w_{,xx}^2 u_{,xx} + 33 u_{,x}^4 w_{,xx} u_{,xxx} \right. \\ \left. - 72 w_{,x} u_{,xx}^2 w_{,xx}^2 u_{,xx} + 4w_{,x}^4 u_{,xx} w_{,xxx} + 72 u_{,x}^2 u_{,xx}^3 w_{,x} - w_{,x}^5 u_{,xxxx} + 16 w_{,x}^3 u_{,x} w_{,xx} w_{,xxx} \right. \\ \left. + 2w_{,x}^4 u_{,xx} w_{,xxxx} + 60 u_{,x}^4 u_{,xx} w_{,xxx} - 48 w_{,x}^2 u_{,xx}^2 u_{,xx} w_{,xxx} + 3w_{,x} u_{,xx}^4 u_{,xxxx} \right. \\ \left. - 3w_{,x}^4 w_{,xx} u_{,xxx} + 2w_{,x}^3 u_{,xx}^2 u_{,xxxx} - 4w_{,x}^3 u_{,xx}^3 - 4w_{,x}^3 u_{,xx} u_{,xx} w_{,xxx} + 60 w_{,x} u_{,xx}^3 u_{,xx} w_{,xxx} \right. \\ \left. - 60 w_{,x}^2 u_{,xx} w_{,xx} u_{,xx}^2 - 18 w_{,x}^2 u_{,xx}^2 w_{,xx} u_{,xxx} - 8u_{,x}^3 w_{,xx}^3 + 12 w_{,x}^2 u_{,xx} w_{,xx}^3 + 6u_{,x}^5 w_{,xxxx} \right. \\ \left. - 8w_{,x}^2 u_{,xx}^3 w_{,xxxx} + 144 u_{,x}^3 w_{,xx} u_{,xx}^2 \right);$$

$$T_{2,7} = EI \left( -w_{,x}^5 u_{,xx} u_{,xxxx} + 45 u_{,x}^5 w_{,xx} u_{,xxx} + w_{,x}^4 u_{,xx}^2 w_{,xxxx} + 108 u_{,x}^5 u_{,xx} w_{,xxx} \right. \\ \left. - 9w_{,x}^5 u_{,xx} u_{,xxx} - 18 w_{,x}^4 w_{,xx} u_{,xx}^2 + 216 u_{,x}^4 u_{,xx}^2 w_{,xx} - 6w_{,x}^2 u_{,xx}^4 w_{,xxxx} - 9w_{,x} u_{,xx}^5 u_{,xxxx} \right. \\ \left. + 6w_{,x}^3 u_{,xx}^3 u_{,xxxx} + 6w_{,x}^2 u_{,xx}^2 w_{,xx}^3 + 60 w_{,x}^3 u_{,xx} u_{,xx}^3 - 216 w_{,x} u_{,xx}^3 u_{,xx}^3 \right. \\ \left. - 48 w_{,x}^2 u_{,xx}^3 u_{,xx} w_{,xxx} - 153 w_{,x} u_{,xx}^4 w_{,xx} u_{,xxx} + 78 w_{,x}^3 u_{,xx}^2 u_{,xx} u_{,xxx} + 4w_{,x}^4 u_{,xx} u_{,xx} w_{,xxx} \right. \\ \left. 18 w_{,x}^2 u_{,xx}^2 w_{,xx} u_{,xx}^2 - 3w_{,x}^4 u_{,xx} w_{,xx} u_{,xxx} - 72 w_{,x} u_{,xx}^3 w_{,xx}^2 u_{,xx} + 12 w_{,x}^3 u_{,xx} w_{,xx}^2 u_{,xx} \right. \\ \left. - 24 w_{,x} u_{,xx}^4 w_{,xx} w_{,xxx} - 6w_{,x}^2 u_{,xx}^3 w_{,xx} u_{,xxx} + 8w_{,x}^3 u_{,xx}^2 w_{,xx} w_{,xxx} - 6u_{,x}^4 w_{,xx}^3 \right. \\ \left. + 9u_{,x}^6 w_{,xxxx} \right);$$

## A.2

### Termos da equação na forma adimensional

$$T_{1,0}^* = 0 ;$$

$$T_{1,1}^* = u_{,\tau\tau}^* + \alpha\beta u_{,\tau}^* + (P^* - \lambda)\alpha^4 u_{,\zeta\zeta}^* ;$$

$$T_{1,2}^* = -\alpha^4 (w_{,\zeta}^* w_{,\zeta\zeta\zeta\zeta}^* + w_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^*) - \kappa\alpha^4 (w_{,\zeta}^* w_{,\zeta\zeta}^* + 3u_{,\zeta}^* u_{,\zeta\zeta}^*) ;$$

$$\begin{aligned} T_{1,3}^* = & \alpha^4 (5w_{,\zeta}^* u_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^* + w_{,\zeta}^{*2} u_{,\zeta\zeta\zeta\zeta}^* + 4w_{,\zeta}^* w_{,\zeta\zeta}^* u_{,\zeta\zeta\zeta}^* \\ & + 2w_{,\zeta\zeta}^{*2} u_{,\zeta\zeta}^* + 3w_{,\zeta}^* u_{,\zeta}^* w_{,\zeta\zeta\zeta\zeta}^* + 3u_{,\zeta}^* w_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^*) ; \\ & - \frac{1}{2} \kappa\alpha^4 (w_{,\zeta}^{*2} u_{,\zeta\zeta}^* + 3u_{,\zeta}^{*2} u_{,\zeta\zeta}^* + 2w_{,\zeta}^* u_{,\zeta}^* w_{,\zeta\zeta}^*) \end{aligned}$$

$$\begin{aligned} T_{1,4}^* = & 2\alpha^4 (2w_{,\zeta}^{*3} w_{,\zeta\zeta\zeta\zeta}^* - 4w_{,\zeta}^{*2} u_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^* - 6w_{,\zeta}^* w_{,\zeta\zeta}^* u_{,\zeta\zeta}^{*2} \\ & - 8w_{,\zeta}^* u_{,\zeta}^* w_{,\zeta\zeta}^* u_{,\zeta\zeta\zeta}^* - 3u_{,\zeta}^{*2} w_{,\zeta}^* w_{,\zeta\zeta\zeta\zeta}^* - 2w_{,\zeta}^{*2} u_{,\zeta}^* u_{,\zeta\zeta\zeta\zeta}^* \\ & + 6w_{,\zeta}^* w_{,\zeta\zeta}^{*3} - 15u_{,\zeta}^{*2} w_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^* - 16u_{,\zeta}^* w_{,\zeta\zeta}^{*2} u_{,\zeta\zeta}^* \\ & + 9w_{,\zeta}^{*2} w_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^* - 10w_{,\zeta}^* u_{,\zeta}^* u_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^*) ; \end{aligned}$$

$$\begin{aligned} T_{1,5}^* = & 2\alpha^4 (-6w_{,\zeta}^* u_{,\zeta}^* w_{,\zeta\zeta}^{*3} - 5w_{,\zeta}^{*3} u_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^* - w_{,\zeta}^{*4} u_{,\zeta\zeta\zeta\zeta}^* \\ & - 8w_{,\zeta}^{*3} w_{,\zeta\zeta}^* u_{,\zeta\zeta\zeta}^* + 28u_{,\zeta}^{*2} w_{,\zeta\zeta}^* u_{,\zeta\zeta}^{*2} - 15w_{,\zeta}^{*2} w_{,\zeta\zeta}^* u_{,\zeta\zeta}^{*2} \\ & + 20w_{,\zeta}^* u_{,\zeta}^{*2} w_{,\zeta\zeta}^* u_{,\zeta\zeta\zeta}^* + w_{,\zeta}^* u_{,\zeta}^{*3} w_{,\zeta\zeta\zeta\zeta}^* + 13w_{,\zeta}^* u_{,\zeta}^{*2} u_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^* \\ & + 5w_{,\zeta}^{*2} u_{,\zeta}^{*2} u_{,\zeta\zeta\zeta\zeta}^* + 5w_{,\zeta}^{*2} u_{,\zeta\zeta}^{*3} - w_{,\zeta}^{*3} u_{,\zeta}^* w_{,\zeta\zeta\zeta}^* - 9w_{,\zeta}^{*2} u_{,\zeta}^* w_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^* \\ & + 30w_{,\zeta}^* u_{,\zeta}^* w_{,\zeta\zeta}^* u_{,\zeta\zeta}^{*2} + 13u_{,\zeta}^{*3} w_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^* + 20w_{,\zeta}^{*2} u_{,\zeta}^* u_{,\zeta\zeta}^* u_{,\zeta\zeta\zeta}^*) ; \end{aligned}$$

$$\begin{aligned} T_{1,6}^* = & \alpha^4 (32w_{,\zeta}^{*3} u_{,\zeta}^* w_{,\zeta\zeta}^* u_{,\zeta\zeta\zeta}^* + 24w_{,\zeta}^{*3} w_{,\zeta\zeta}^* u_{,\zeta\zeta}^{*2} + 36w_{,\zeta}^* u_{,\zeta}^{*2} w_{,\zeta\zeta}^{*3} \\ & - 108w_{,\zeta}^* u_{,\zeta}^{*2} w_{,\zeta\zeta}^* u_{,\zeta\zeta}^{*2} - 17w_{,\zeta}^{*4} w_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^* - 72u_{,\zeta}^{*3} w_{,\zeta\zeta}^* u_{,\zeta\zeta}^{*2} - w_{,\zeta}^{*5} w_{,\zeta\zeta\zeta\zeta}^* \\ & - 48w_{,\zeta}^* u_{,\zeta}^{*3} w_{,\zeta\zeta}^* u_{,\zeta\zeta\zeta\zeta}^* + 3w_{,\zeta}^* u_{,\zeta}^{*4} w_{,\zeta\zeta\zeta\zeta}^* + 8w_{,\zeta}^{*4} u_{,\zeta\zeta}^* u_{,\zeta\zeta\zeta}^* \\ & + 4w_{,\zeta}^{*4} u_{,\zeta}^* u_{,\zeta\zeta\zeta\zeta}^* - 12w_{,\zeta}^{*2} u_{,\zeta}^{*3} u_{,\zeta\zeta\zeta\zeta}^* - 24w_{,\zeta}^{*3} w_{,\zeta\zeta}^{*3} + 20w_{,\zeta}^{*3} u_{,\zeta}^* u_{,\zeta\zeta}^* w_{,\zeta\zeta\zeta}^*) \end{aligned}$$



$$+84w_{,\zeta}^{*2}u_{,\zeta}^{*}w_{,\zeta\zeta}^{*2}u_{,\zeta\zeta}^{*}+2w_{,\zeta}^{*3}u_{,\zeta}^{*2}w_{,\zeta\zeta\zeta}^{*}+42w_{,\zeta}^{*2}u_{,\zeta}^{*2}w_{,\zeta\zeta}^{*}w_{,\zeta\zeta}^{*};$$

$$-12w_{,\zeta}^{*}u_{,\zeta}^{*3}w_{,\zeta\zeta}^{*}-72w_{,\zeta}^{*2}u_{,\zeta}^{*2}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*});$$

$$T_{1,7}^{*}=EI\left(-w_{,\zeta}^{*5}u_{,\zeta}^{*}w_{,\zeta\zeta\zeta\zeta}^{*}-81u_{,\zeta}^{*5}w_{,\zeta\zeta}^{*}w_{,\zeta\zeta}^{*}-162u_{,\zeta}^{*4}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*}\right.$$

$$+5w_{,\zeta}^{*5}u_{,\zeta\zeta}^{*}w_{,\zeta\zeta}^{*}+12w_{,\zeta}^{*5}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*}+24w_{,\zeta}^{*4}u_{,\zeta\zeta}^{*}w_{,\zeta\zeta}^{*2}+9w_{,\zeta}^{*2}u_{,\zeta}^{*4}w_{,\zeta\zeta\zeta\zeta}^{*}$$

$$-6w_{,\zeta}^{*4}u_{,\zeta}^{*2}w_{,\zeta\zeta\zeta\zeta}^{*}-9w_{,\zeta}^{*}u_{,\zeta}^{*5}w_{,\zeta\zeta\zeta\zeta}^{*}+6w_{,\zeta}^{*3}u_{,\zeta}^{*3}w_{,\zeta\zeta\zeta\zeta}^{*}+54w_{,\zeta}^{*2}u_{,\zeta}^{*2}w_{,\zeta\zeta}^{*3}$$

$$+60w_{,\zeta}^{*}w_{,\zeta\zeta}^{*3}u_{,\zeta}^{*3}-24w_{,\zeta}^{*3}u_{,\zeta}^{*}w_{,\zeta\zeta}^{*3}-48w_{,\zeta}^{*3}u_{,\zeta}^{*2}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*}$$

$$+36w_{,\zeta}^{*}u_{,\zeta}^{*4}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*}+78w_{,\zeta}^{*2}u_{,\zeta}^{*3}w_{,\zeta\zeta}^{*}w_{,\zeta\zeta}^{*}-17w_{,\zeta}^{*4}u_{,\zeta}^{*}w_{,\zeta\zeta}^{*}w_{,\zeta\zeta}^{*}$$

$$18w_{,\zeta}^{*2}u_{,\zeta}^{*2}w_{,\zeta\zeta}^{*2}u_{,\zeta\zeta}^{*}+36w_{,\zeta}^{*}u_{,\zeta}^{*4}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*}-72w_{,\zeta}^{*3}u_{,\zeta}^{*}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*2}$$

$$+108w_{,\zeta}^{*}u_{,\zeta}^{*3}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*2}-24w_{,\zeta}^{*4}u_{,\zeta}^{*}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*}+72w_{,\zeta}^{*2}u_{,\zeta}^{*3}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*}$$

$$\left.-6w_{,\zeta}^{*3}u_{,\zeta}^{*2}w_{,\zeta\zeta}^{*}w_{,\zeta\zeta}^{*}-27w_{,\zeta}^{*}u_{,\zeta}^{*4}w_{,\zeta\zeta}^{*}w_{,\zeta\zeta}^{*}-6w_{,\zeta}^{*4}u_{,\zeta}^{*3}+w_{,\zeta}^{*6}u_{,\zeta\zeta\zeta\zeta}^{*}\right);$$

$$T_{2,0}^{*}=X_0^{*}\cos(\tau);$$

$$T_{2,1}^{*}=w_{,\tau\tau}^{*}+\alpha^2\beta u_{,\tau}^{*}+\lambda\alpha^4w_{,\zeta\zeta}^{*}-\alpha^4w_{,\zeta\zeta\zeta\zeta}^{*};$$

$$T_{2,2}^{*}=-\alpha^4(w_{,\zeta}^{*}u_{,\zeta\zeta\zeta\zeta}^{*}+4u_{,\zeta\zeta}^{*}w_{,\zeta\zeta\zeta}^{*}+3w_{,\zeta\zeta}^{*}u_{,\zeta\zeta\zeta}^{*}+2u_{,\zeta}^{*}w_{,\zeta\zeta\zeta\zeta}^{*});$$

$$-\kappa\alpha^4(w_{,\zeta}^{*}u_{,\zeta\zeta}^{*}+u_{,\zeta}^{*}w_{,\zeta\zeta}^{*});$$

$$T_{2,3}^{*}=\alpha^4\left(7w_{,\zeta}^{*}u_{,\zeta\zeta}^{*}u_{,\zeta\zeta\zeta}^{*}-2w_{,\zeta}^{*2}w_{,\zeta\zeta\zeta\zeta}^{*}-8w_{,\zeta}^{*}w_{,\zeta\zeta}^{*}w_{,\zeta\zeta\zeta}^{*}+8w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*2}\right.$$

$$+3w_{,\zeta}^{*}u_{,\zeta}^{*}u_{,\zeta\zeta\zeta\zeta}^{*}+9u_{,\zeta}^{*}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta\zeta}^{*}+12u_{,\zeta}^{*}u_{,\zeta\zeta}^{*}w_{,\zeta\zeta\zeta}^{*}+3u_{,\zeta}^{*2}w_{,\zeta\zeta\zeta\zeta}^{*}-2w_{,\zeta\zeta}^{*3}$$

$$\left.-\frac{1}{2}\alpha^4\kappa\left(u_{,\zeta}^{*2}w_{,\zeta\zeta}^{*}+3w_{,\zeta}^{*2}w_{,\zeta\zeta}^{*}+2w_{,\zeta}^{*}u_{,\zeta}^{*}u_{,\zeta\zeta}^{*}\right)\right);$$

$$T_{2,4}^{*}=2\alpha^4\left(w_{,\zeta}^{*3}u_{,\zeta\zeta\zeta\zeta}^{*}+3u_{,\zeta}^{*2}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta\zeta}^{*}+8u_{,\zeta}^{*}w_{,\zeta\zeta}^{*}u_{,\zeta\zeta}^{*2}-14w_{,\zeta}^{*}u_{,\zeta}^{*}u_{,\zeta\zeta}^{*}u_{,\zeta\zeta\zeta}^{*}\right.$$

$$-3u_{,\zeta}^{*2}w_{,\zeta}^{*}u_{,\zeta\zeta\zeta\zeta}^{*}+3w_{,\zeta}^{*2}u_{,\zeta\zeta}^{*}u_{,\zeta\zeta\zeta}^{*}-4w_{,\zeta}^{*}u_{,\zeta\zeta}^{*3}+12u_{,\zeta}^{*2}u_{,\zeta\zeta}^{*}w_{,\zeta\zeta\zeta}^{*}$$

$$\left.+2u_{,\zeta}^{*3}w_{,\zeta\zeta\zeta\zeta}^{*}\right);$$

$$T^*_{2,5} = \alpha^4 \left( -8w^*_{,\zeta} u^*_{,\zeta} u^{*3}_{,\zeta\zeta} - +8w^{*3}_{,\zeta} w^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} + w^{*4}_{,\zeta} w^*_{,\zeta\zeta\zeta\zeta} + 2w^{*3}_{,\zeta} u^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} \right. \\ + 6w^{*2}_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta} - 64u^{*2}_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta} - 2w^*_{,\zeta} u^{*2}_{,\zeta} u^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} - 2w^{*3}_{,\zeta} u^*_{,\zeta} u^*_{,\zeta\zeta\zeta} \\ - 6w^{*2}_{,\zeta} u^*_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} + 2w^*_{,\zeta} u^{*3}_{,\zeta} u^*_{,\zeta\zeta\zeta} + 6w^{*2}_{,\zeta} w^*_{,\zeta\zeta} - 18u^{*3}_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} \\ \left. - 5u^{*4}_{,\zeta} w^*_{,\zeta\zeta\zeta\zeta} - 40u^{*3}_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} \right);$$

$$T^*_{2,6} = \alpha^4 \left( -32w^*_{,\zeta} u^{*3}_{,\zeta} w^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} + 12w^{*3}_{,\zeta} w^{*2}_{,\zeta\zeta} u^*_{,\zeta\zeta} + 33u^{*4}_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} \right. \\ - 72w^*_{,\zeta} u^{*2}_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta} + 4w^{*4}_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} + 72u^{*2}_{,\zeta} u^{*3}_{,\zeta\zeta} w^*_{,\zeta} - w^{*5}_{,\zeta} u^*_{,\zeta\zeta\zeta\zeta} \\ + 16w^{*3}_{,\zeta} u^*_{,\zeta} w^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} - 3w^{*4}_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta} + 12w^{*2}_{,\zeta} u^*_{,\zeta} w^{*3}_{,\zeta\zeta} \zeta \\ + 2w^{*4}_{,\zeta} u^*_{,\zeta} w^*_{,\zeta\zeta\zeta\zeta} + 60u^{*4}_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} - 48w^{*2}_{,\zeta} u^{*2}_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} + 3w^*_{,\zeta} u^{*4}_{,\zeta} u^*_{,\zeta\zeta\zeta\zeta} \\ + 2w^{*3}_{,\zeta} u^{*2}_{,\zeta} u^*_{,\zeta\zeta\zeta\zeta} - 4w^{*3}_{,\zeta} u^{*3}_{,\zeta\zeta} - 4w^{*3}_{,\zeta} u^*_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} + 60w^*_{,\zeta} u^{*3}_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} \\ - 60w^{*2}_{,\zeta} u^*_{,\zeta} w^*_{,\zeta\zeta} u^{*2}_{,\zeta\zeta} - 18w^{*2}_{,\zeta} u^{*2}_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} - 8u^{*3}_{,\zeta} w^{*3}_{,\zeta\zeta} + 6u^{*5}_{,\zeta} w^*_{,\zeta\zeta\zeta\zeta} \\ \left. - 8w^{*2}_{,\zeta} u^{*3}_{,\zeta} w^*_{,\zeta\zeta\zeta\zeta} + 144u^{*3}_{,\zeta} w^*_{,\zeta\zeta} u^{*2}_{,\zeta\zeta} \right);$$

$$T^*_{2,7} = \alpha^4 \left( -w^{*5}_{,\zeta} u^*_{,\zeta} u^*_{,\zeta\zeta\zeta\zeta} + 45u^{*5}_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} + w^{*4}_{,\zeta} u^{*2}_{,\zeta} w^*_{,\zeta\zeta\zeta\zeta} \right. \\ + 108u^{*5}_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} - 9w^*_{,\zeta} u^{*5}_{,\zeta} u^*_{,\zeta\zeta\zeta\zeta} + 4w^{*4}_{,\zeta} u^*_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} \\ - 9w^{*5}_{,\zeta} u^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} - 18w^{*4}_{,\zeta} w^*_{,\zeta\zeta} u^{*2}_{,\zeta\zeta} + 216u^{*4}_{,\zeta} u^{*2}_{,\zeta\zeta} w^*_{,\zeta\zeta} - 6w^{*2}_{,\zeta} u^{*4}_{,\zeta} w^*_{,\zeta\zeta\zeta\zeta} \\ + 6w^{*3}_{,\zeta} u^{*3}_{,\zeta} u^*_{,\zeta\zeta\zeta\zeta} + 6w^{*2}_{,\zeta} u^{*2}_{,\zeta} w^{*3}_{,\zeta\zeta} + 60w^{*3}_{,\zeta} u^*_{,\zeta} u^{*3}_{,\zeta\zeta} - 216w^*_{,\zeta} u^{*3}_{,\zeta} u^{*3}_{,\zeta\zeta} \\ - 48w^{*2}_{,\zeta} u^{*3}_{,\zeta} u^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta\zeta} - 153w^*_{,\zeta} u^{*4}_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} + 78w^{*3}_{,\zeta} u^{*2}_{,\zeta} u^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} \\ 18w^{*2}_{,\zeta} u^{*2}_{,\zeta} w^*_{,\zeta\zeta} u^{*2}_{,\zeta\zeta} - 3w^{*4}_{,\zeta} u^*_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} - 72w^*_{,\zeta} u^{*3}_{,\zeta} w^{*2}_{,\zeta\zeta} u^*_{,\zeta\zeta} \\ - 24w^*_{,\zeta} u^{*4}_{,\zeta} w^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} - 6w^{*2}_{,\zeta} u^{*3}_{,\zeta} w^*_{,\zeta\zeta} u^*_{,\zeta\zeta\zeta} + 8w^{*3}_{,\zeta} u^{*2}_{,\zeta} w^*_{,\zeta\zeta} w^*_{,\zeta\zeta\zeta} \\ \left. - 6u^{*4}_{,\zeta} w^{*3}_{,\zeta\zeta} + 12w^{*3}_{,\zeta} u^*_{,\zeta} w^{*2}_{,\zeta\zeta} u^*_{,\zeta\zeta} + 9u^{*6}_{,\zeta} w^*_{,\zeta\zeta\zeta\zeta\zeta} \right);$$

## Apêndice B

### Algoritmo utilizando método do controle do comprimento de arco

O presente apêndice tem como objetivo principal descrever os comandos do Programa MAPLE9, utilizado para a resolução do sistema de equações algébricas não-lineares, utilizando o método do controle de comprimento de arco. Como exemplo, utiliza-se a vibração forçada amortecida de uma viga biapoada.

#### B.1

##### Algoritmo

```
> ###Algoritmo utilizando o método do controle de comprimento de arco
para obtenção da curva de ressonância para uma viga biapoada###
```

```
> restart:
```

```
> with(linalg):
```

```
> with(plots):
```

```
> Digits:=50:
```

```
> interface(displayprecision=5):
```

```
#Inicializa variáveis do Problema
```

```
> eta:=0.5:# Parâmetro igual a  $(h \cdot \pi / L)^2$ 
```

```
> X0:=0.5:# Amplitude adimensional da carga co-senoidal aplicada
```

```
> xi:=0.025:# Fator de atrito
```

```
> beta:=2*xi:# Constante de atrito adimensional
```

```
> DL:=0.01:# Comprimento do arco
```

```
> .DL2:=DL*DL: # Comprimento do arco ao quadrado
```

```
> psi:=0.01: # fator de escala utilizado no método do comprimento de arco
```

```
#Rotinas
```

```
#Rotina para calcular a função resíduo igual a equação de equilíbrio num
ponto (X, lambda*omega)
```

```
# x – vetor de amplitudes
```

```
#omega- frequência
```

```
#lambda – fator de frequência
```

```

> calcr:=proc(eta,X0,beta,lambda,omega,X)
>   global r:
>   local rl:
>
>   rl[1]:=evalf(-
4*X0*omega^2*lambda^2*Pi*X[1]+3/8*eta*Pi*X[1]^3+15/256*eta^2*Pi*X[
1]^5+Pi*X[1]+15/256*eta^2*Pi*X[1]*X[2]^4+15/128*eta^2*Pi*X[1]^3*X[2]
^2+3/8*eta*Pi*X[1]*X[2]^2+omega*lambda*beta*Pi*X[2]);
>   rl[2]:=evalf(Pi*X[2]-
omega^2*lambda^2*Pi*X[2]+3/8*eta*Pi*X[2]^3+15/256*eta^2*Pi*X[2]^5+1
5/128*eta^2*Pi*X[1]^2*X[2]^3+3/8*eta*Pi*X[1]^2*X[2]+15/256*eta^2*Pi*
X[1]^4*X[2]-omega*lambda*beta*Pi*X[1]):r:=matrix((2,1,[rl[1],rl[2]])):
> end proc:

```

#Rotina para calcular a derivada de r em relação a X em um ponto (X, lambda\*omega)

```

> calcKt:=proc(eta,X0,beta,lambda,omega,X)
>   global Kt:
>   local K:
>   K[1,1]:=evalf(-
omega^2*lambda^2*Pi+9/8*eta*Pi*X[1]^2+75/256*eta^2*Pi*X[1]^4+Pi+15/
256*eta^2*Pi*X[2]^4+45/128*eta^2*Pi*X[1]^2*X[2]^2+3/8*eta*Pi*X[2]^2);
> K[1,2]:=evalf(15/64*eta^2*Pi*X[1]*X[2]^3+15/64*eta^2*Pi*X[1]^3*X
[2]+3/4*eta*Pi*X[1]*X[2]+omega*lambda*beta*Pi);
>
> K[2,1]:=evalf(15/64*eta^2*Pi*X[1]*X[2]^3+3/4*eta*Pi*X[1]*X[2]+15/
64*eta^2*Pi*X[1]^3*X[2]-omega*lambda*beta*Pi);
> K[2,2]:=evalf(Pi*omega^2*lambda^2*Pi+9/8*eta*Pi*X[2]^2+75/256*
eta^2*Pi*X[2]^4+45/128*eta^2*Pi*X[1]^2*X[2]^2+3/8*eta*Pi*X[1]^2
+15/256*eta^2*Pi*X[1]^4);
> Kt:=matrix(2,2,[[K[1,1],K[1,2]],[K[2,1],K[2,2]]]):
> end proc:

```

Rotina para calcular a derivada de r em relação a lambda em um ponto (X, lambda\*omega)

```

> calcq:=proc(eta,X0,beta,lambda,omega,X)

```

```

>      global q:
>      local ql:
>      ql[1]:=evalf(2*omega^2*lambda*Pi*X[1]+omega*beta*Pi*X[2]):
>      ql[2] :=evalf(-2*omega^2*lambda*Pi*X[2]-omega*beta*Pi*X[1]):
>      q:=matrix(2,1,[ql[1],ql[2]]):
> end proc:

```

#Rotina que calcula o preditor para o incremento do fator de frequência (Deltalambda) para um comprimento de arco dado (Dl) e um fator de escala escolhido psi (psi=0, comprimento de arco cilíndrico)

```

> dlambdainic:=proc(eta,X0,beta,lambda,omega,X,psi,Dl)
>      global Dlambda0,df;
>      local deter,Kq,Kq2:
>      Kq:=matrix(2,1):Kq2:=matrix(1,1):
>      calcq(eta,X0,beta,lambda,omega,X):
>      calcKt(eta,X0,beta,lambda,omega,X):
>      deter:=det(Kt):
>
>      df:=inverse(Kt):Kq:=multiply(df,q):Kq2:=multiply((transpose(Kq)),K
q):
>Dlambda0:=sign(deter)*abs(Dl/(sqrt((Kq2[1,1])^2+psi^2*lambda^2*omega
^2))):
> end proc:

```

#Rotina para Resolver uma equação quadrática  $A_1X^2 + A_2X + A_3$  -  
#e fornecer IFAIL  
#IFAIL 0 - duas raízes complexas  
#IFAIL 1 - A1 muito pequeno, solução linear, uma só raiz  
#IFAIL 2 - duas raízes reais

```

> QSOLV:=proc(A1,A2,A3)
>      local SMALL,FAC:
>      global R1,R2,RLIN, IFAIL:
>      SMALL:=1.*10.^(-10):
>      IFAIL:=0:
>      if (A2 <> 0.) then
>          RLIN:=-A3/A2:

```

```

>      fi:
>      FAC:=A2*A2-4.*A1*A3:
>      if (FAC < 0.) then
#Não tem raízes reais
>          IFAIL:=0:
>      else
#Raízes reais
>          FAC:=sqrt(FAC):
>          if (A1 = 0.) then
>              if (A2 <> 0.) then
>                  R1:=RLIN:
>                  IFAIL:=1:
>              else
>                  stop;
>          fi:
>      else
>          R1:=-0.5*(FAC+A2)/A1:
>          R2:=0.5*(FAC-A2)/A1:
>      fi:
>      fi:
> end proc:

```

#Rotina que calcula numa iteração do processo corretor um ponto  $(X[k+1], \lambda[k+1] \cdot \omega)$  a partir de um ponto  $(X[k], \lambda[k] \cdot \omega)$  e um preditor

#Entrada:

#DT - Deslocamento correspondente a uma frequência fixa  $\omega$ , solução tangencial

#DELBAR - Deslocamento a um nível fixo de frequência  $\Delta \lambda$

#DX - Incremento de Amplitude

#DL2 - DL ao quadrado

#DLAMBDA - Fator de frequência total

#Saída:

#FACT - Novo fator de frequência total

#PT - Nova amplitude

```

> ARCL1:= proc(DT,DELBAR,DX,DL2,DLAMBDA,psi,omega)
>     local i,COST1,COST2,SOL,A1,A2,A3,A4,A5,DPBAR:
>     global FACT,PT:
>     A1:=psi^2*omega^2:
>     A2:=2.*DLAMBDA*psi^2*omega^2:
>     A3:=-DL2+DLAMBDA^2*psi^2*omega^2:
>     A4:=0.:
>     A5:=0.:
>     for i from 1 to 2 do
>         A1:= A1+DT[i,1]*DT[i,1]:
>         DPBAR[i]:=DX[i,1]-DELBAR[i,1]:
>         A2:=A2-2.*DT[i,1]*DPBAR[i]:
>         A3:=A3+DPBAR[i]*DPBAR[i]:
>         A4:=A4+DPBAR[i]*(DX[i,1]):
>         A5:=A5- DT[i,1]*(DX[i,1]):
>     od:
>     QSOLV(A1,A2,A3):
>     if (IFAIL = 2) then return: fi:
>     if (IFAIL = 1) then
>         SOL:= R1:
>     else
>         COST1:=A4+A5*R1:
>         COST2:=A4+A5*R2:
>         SOL:=R1:
>         if (COST2 > COST1) then
>             SOL:=R2:
>         fi:
>     fi:
>     FACT:=DLAMBDA+SOL:
>     for i from 1 to 2 do
>         PT[i,1]:=DX[i,1]-DELBAR[i,1]-SOL*DT[i,1]:
>     od:
> end proc:

```

#Encontrar Ponto Inicial com frequência próximo a zero - correspondendo a uma solução inicial

```
>X1:=vector(2):X:=vector(2):dX:=matrix(2,1):DXp:=matrix(2,1):df:=matrix(2,2):DELBAR:=matrix(2,1):DT:=matrix(2,1):PT:=matrix(2,1):
```

```
> omega:=0.1: lambda1:=1.:X1[1]:=0.:X1[2]:=0.:
```

#Ponto inicial para o processo iterativo

#Newton- Raphson com frequência constante - até convergir para X1 – tol = tolerância

```
> niter:=0:erro:=100.:tol:=0.00000001:
```

```
> while (niter<=20) and (erro>tol) do
```

```
>     niter:=niter+1:
```

```
>calcr(etaX0,beta,lambda1,omega,X1):calcKt(eta,X0,beta,lambda1,omega,X1):
```

```
>     df:=inverse(Kt):
```

```
>     dX:=multiply(df,r):
```

```
>     X1[1]:=X1[1]-dX[1,1]:X1[2]:=X1[2]-dX[2,1]:
```

```
>     erro:=abs(evalf(sqrt(
        ((dX[1,1])/X1[1])^2+((dX[2,1])/X1[2])^2))):
```

```
> od:
```

```
print(X1,erro):
```

# Incremento, preditor e corretor

```
> Nmax:=300:nincrp:=0:tniter:=0:niterd:=5:
```

```
while (nincrp<Nmax) do
```

```
>     nincrp:=nincrp+1:
```

```
> dlambdainic(etaX0,beta,lambda1,omega,X1,psi,DL):
```

```
> calcKt(eta,X0,beta,lambda1,omega,X1):df:=inverse(Kt):
```

```
>     calcq(eta,X0,beta,lambda1,omega,X1):
```

```
>     DXp:=(multiply(df,q)):
```

#Cálculo da direção do incremento de amplitude

```
>     dX[1,1]:=-Dlambda0*DXp[1,1]:dX[2,1]:=Dlambda0*DXp[2,1]:
```

#Resultado da Fase preditora

```
>     X[1]:=X1[1]+dX[1,1]:X[2]:=X1[2]+dX[2,1]:
```

```
>     lambda:=lambda1+Dlambda0:
```

```
>     Dlambda:=Dlambda0:
```



```

>      niter:=0: tol:=.000000001:erro:=100.:
#Iteracoes para correção
>      while (niter <20) and (erro > tol) do
>          niter:=niter+1:
>          calcr(eta,X0,beta,lambda,omega,X):
>          calcKt(eta,X0,beta,lambda,omega,X):df:=inverse(Kt):
>          calcq(eta,X0,beta,lambda,omega,X):
>          DELBAR:=(multiply(df,r)):
>          DT:=(multiply(df,q)):DL2:=DL*DL:
>          ARCL1(DT,DELBAR,dX,DL2,Dlambda,psi,omega):
>          dX[1,1]:=PT[1,1]:dX[2,1]:=PT[2,1]:
>          X[1]:=X1[1]+dX[1,1]:X[2]:=X1[2]+dX[2,1]:
>          Dlambda:=FACT:lambda:=lambda1+Dlambda:
>          calcr(eta,X0,beta,lambda,omega,X):
>          erro:=abs(sqrt(r[1,1]^2+r[2,1]^2)):
>      od:
>      X1[1]:=X[1]:X1[2]:=X[2]:lambda1:=lambda:
>      XGRAFp[nincrp]:=sqrt(h^2*X[1]^2+h^2*X[2]^2):
>      WGRAFp[nincrp]:=omega*lambda:DLGRAFp[nincrp]:=DL:
>      X1GRAF[nincrp]:=h*X[1]: X2GRAF[nincrp]:=h*X[2]:
>      tniter:=tniter+niter:
> od:
#Plotar curvas
>a:=plot({[seq([WGRAFp[i],XGRAFp[i]],i=1.. nincrp)]},color='black'):
> c:=plot({[seq([WGRAFp[i],X1GRAF[i]],i=1..nincrp)]},color='black'):
> d:=plot({[seq([WGRAFp[i],X2GRAF[i]],i=1..nincrp)]},color='black'):
> display(a);
> display(c);
> display(d);

```

## Anexo I

### Formulação em elementos finitos

Neste anexo apresentam-se as funções de forma utilizadas para a obtenção das matrizes de massa  $M$  e rigidez elástica  $K$  e rigidez geométrica  $K_g$  para vigas e pórticos.

#### I.1

##### Elementos de vigas

Desprezando-se os efeitos de deformação por esforços cortante e axial, as funções de forma cúbicas  $N(x)$ , são dadas por:

$$N_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3} \quad (\text{I-1})$$

$$N_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2} \quad (\text{I-2})$$

$$N_3(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} \quad (\text{I-3})$$

$$N_4(x) = -\frac{x^2}{L^2} + \frac{x^3}{L^2} \quad (\text{I-4})$$

Interpolando-se os deslocamentos no elemento pelos deslocamentos nos nós ( $u_1, u_2, u_3$  e  $u_4$ ) e utilizando-se as funções de forma acima, chega-se a matriz de massa  $M$ :

$$M = \frac{\rho AL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \quad (\text{I-5})$$

E a matriz de rigidez elástica  $K$ :

$$K = \frac{EI}{L} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (\text{I-6})$$

Detalhes da derivação das matrizes aqui apresentadas encontram-se no trabalho de Meirovitch (1975).

## I.2

### Elementos de viga-coluna

As funções de forma  $N(x)$  para elementos de viga-coluna, utilizadas para análise de pórticos planos pelo método dos elementos finitos, são dadas por:

$$N_1(x) = 1 - \frac{x}{L}; \quad (\text{I-7})$$

$$N_2(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}; \quad (\text{I-8})$$

$$N_3(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2}; \quad (\text{I-9})$$

$$N_4(x) = \frac{x}{L}; \quad (\text{I-10})$$

$$N_5(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}; \quad (\text{I-11})$$

$$N_6(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}; \quad (\text{I-12})$$

Interpolando-se os deslocamentos no elemento pelos deslocamentos nos nós ( $u_1, u_2, u_3, u_4, u_5$  e  $u_6$ ) e utilizando-se as funções de forma acima, chega-se a matriz de massa  $M$ :

$$M = \frac{\rho AL}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & 13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \quad (\text{I-13})$$

E a matriz de rigidez elástica e geométrica, são respectivamente:

$$K = \frac{EI}{L^3} \begin{bmatrix} AL^2/I & 0 & 0 & -AL^2/I & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -AL^2/I & 0 & 0 & AL^2/I & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \quad (\text{I-14})$$

$$Kg = \frac{P}{30L} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3L & 0 & -36 & 3L \\ 0 & 3L & 4L^2 & 0 & -3L & -L^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3L & 0 & 36 & -3L \\ 0 & 3L & -L^2 & 0 & -3L & 4L^2 \end{bmatrix} \quad (\text{I-15})$$

Detalhes da derivação das matrizes para análise de pórticos aqui apresentadas encontram-se no trabalho de Paz (1997).

## Anexo II

### Relações trigonométricas

Neste anexo são apresentadas as relações trigonométricas (Beyer, 1987) utilizadas no método do balanço harmônico para linearizar as potências e produtos de co-seno e seno resultantes da aplicação deste método.

$$\text{sen}^{2n}(x) = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} \cos[(n-k)x]; \quad (\text{II-1})$$

$$\cos^{2n}(x) = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{+1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos[(n-k)x]; \quad (\text{II-2})$$

$$\text{sen}^{2n+1}(x) = \frac{(-1)^n}{4^n} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} \text{sen}[(2n+1-2k)x]; \quad (\text{II-3})$$

$$\cos^{2n+1}(x) = \frac{1}{4^n} \sum_{k=0}^n \binom{2n+1}{k} \cos[(2n+1-2k)x]; \quad (\text{II-4})$$

$$\cos(x)\cos(2nx) = \frac{1}{2} \cos[(2n+1)x] + \frac{1}{2} \cos[(2n-1)x]; \quad (\text{II-5})$$

$$\text{sen}(x)\cos(2nx) = \frac{1}{2} \cos[(2n+1)x] - \frac{1}{2} \cos[(2n-1)x] \quad (\text{II-6})$$

Nas expressões acima os símbolos  $\binom{n}{k}$  representam os coeficientes

binomiais, calculados explicitamente pela expressão:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad (\text{II-7})$$

onde  $z!$  denota o fatorial de  $z$ .