6
A dichotomy about $(s,1,u)$-partially hyperbolic attractors

In this section we present an important class of dynamics where u or s-minimality holds. The proof of next theorem is the main goal of this section.

Theorem 6.1 *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$ with the following property. Let $f \in \mathcal{R}$ and $\Lambda_f(U)$ be a transitive proper attractor of f that is $(s,1,u)$-partially hyperbolic, and $\mathcal{U}$ be a compatible neighborhood of f. Let $\mathcal{G}_s$ (resp. $\mathcal{G}_u$) be the subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms g such that $\Lambda_g(U)$ is s-minimal (resp. u-minimal). Then $\mathcal{G}_s \cup \mathcal{G}_u$ is a residual subset of $\mathcal{U}$.*

Corollary 6.2 *Let $\mathcal{R}$, $f \in \mathcal{R}$, and $\Lambda_f(U)$ be as in Theorem 6.1. If $\Lambda_f(U)$ is robustly transitive, then $\mathrm{int}(\mathcal{G}_s) \cup \mathrm{int}(\mathcal{G}_u)$ is an open and dense subset of $\mathcal{U}$.*

Proof: This corollary is an immediate consequence of Corollary 5.17 together with Theorem 6.1 when $\Lambda_f(U)$ is a robustly transitive set. ∎

Corollary 6.3 *Let $f \in \mathcal{R}$ and $\Lambda_f(U)$ be a transitive attractor of f that is $(s,1,u)$-partially hyperbolic. Then for every periodic point $p \in \Lambda_f(U)$ there is an open set $\mathcal{W}_p \subset \mathrm{Diff}^1(M)$, with $f \in \overline{\mathcal{V}_p}$, such that if $g \in \mathcal{V}$ and $\Lambda_g(U)$ is transitive then $\Lambda_g(U) = H(p_g, g)$.*

Proof: By Remark 3.7, any attractor contains its unstable manifolds. In particular, the attractors contain its homoclinic classes, since they are subsets of the unstable manifolds. Then, $H(p_g, g) \subset \Lambda_g(U)$. The inverse inclusion is just Theorem 5.9, giving the equality $\Lambda_g(U) = H(p_g, g)$ in the open set $\mathcal{W}_p$. ∎

Corollary 6.4 *C^1-generically, a robustly transitive attractor $\Lambda_f(U)$ of a diffeomorphism f varies continuously in a small neighborhood of f.*

Proof: This is an immediate consequence of Remark 2.2 and 4.2. Since the attractor is also robustly a homoclinic class, it must vary both lower and upper semicontinously in a open neighborhood of f. ∎

To prove Theorem 6.1 we need to classify the behaviour of the dynamics on certain central invariant curves. According to this classification, the proof of Theorem 6.1 is divided case by case.

6.1
Central Curves: Classification of Periodic Points

Unlike the strong stable and unstable bundles, we can not guarantee the existence of invariant central foliations tangent to the central bundle of a partially hyperbolic splitting. Nevertheless, if Λ is a $(s, 1, u)$-partially hyperbolic attractors, we can guarantee the existence of *invariant central curves* for the hyperbolic periodic points of Λ (Proposition 6.5). A *central curve* is a curve $\gamma \subset M$ that is tangent to the (extended) central bundle E^c at every point of U (see subsection 4.3).

Next result is an adaptation of Theorem 2 in (22) for the context of partially hyperbolic attractors. In the original statement in (22), the partial hyperbolicity is defined in the whole manifold.

Proposition 6.5 *Let $f \in \mathcal{R}$ and $\Lambda_f(U)$ be an $(s, 1, u)$-partially hyperbolic attractor of f. Then there exists $K > 0$ such that, for every hyperbolic periodic point with period $N \geq K$, there exists an f^N invariant central curve $L(p)$ (i.e., $f^N(L(p)) = L(p)$) containing p in its interior.*

ProofSketch of the proof: The proof of this version is exactly the same as the one in Theorem 2 of (22). We only observe that it involves only local arguments, which are still valid inside the isolating block U of the attractor $\Lambda_f(U)$. Then for each periodic point $p \in \Lambda_f(U)$ we obtain a local central curve $\gamma(p)$ inside U. In the process we may assume that either $\gamma(p)$ is a complete curve inside U or its boundary lies in the boundary of U. In the later case, we can still extend $\gamma(p)$ to a complete curve by taking the backward and forward iterations of it. So we define $L(p) = \bigcup_{n\in\mathbb{Z}} f^{n\cdot N}(\gamma(p))$, where N is the period of p. In this way we obtain the invariance by the period $f^N(L(p)) = L(p)$. ■

Let f be a diffeomorphism in the residual subset $\mathcal{R}$ and $\Lambda_f(U)$ be a transitive attractor of f. For each periodic point $p \in \Lambda_f(U)$ of sufficiently large period we consider an invariant central curve $L(p)$ passing through p given by Proposition 6.5. Given $\varepsilon > 0$, we denote by $L_{\varepsilon,U}(p)$ the connected component of $L_U(p) \cap B_\varepsilon(p)$ that contains p.

The rest of this section is a translation, to the case of attractors, of the results in Section 5.2 of (12).

Remark 6.6 Recall that $\mathcal{R}$ consists of Kupka-Smale diffeomorphisms, so the set of periodic points with period less than a given constant K is a finite set. Note that if the period of p is d, then the period of any periodic point in the curve $L(p)$ is a divisor of $2d$ (the factor 2 appears because f may reverse the orientation in the central direction). Hence, there are only finitely many periodic points in $L(p)$.

We choose these central curves $L(p)$ in a coherent way, that is, satisfying $f(L(p)) = L(f(p))$.

Denote by $L_U(p)$ the connected component of $L(p) \cap U$ containing p and let $\Gamma_p \subset L_U(p)$ be the smallest compact and connected subset of $L_U(p)$ that contains all periodic points and all periodic closed curves of $L_U(p)$ (it may happens that $\Gamma_p = \{p\}$). There are three possibilities for the boundary $\partial\Gamma_p$ of Γ_p relative to the set $L_U(p)$: either it is an empty set, a unitary set, or a two points set. If $\partial\Gamma_p = \emptyset$ then Γ_p is a closed curve. When $\partial\Gamma_p \neq \emptyset$ we say that $\partial\Gamma_p$ are the *extremal points* of Γ_p. A periodic point q is called *extremal* if there is some $p \in \Lambda_f(U)$ such that $q \in \partial\Gamma_p$.

Remark 6.7 Since U is a neighborhood of the compact set $\Lambda_f(U)$, the length of $L_U(p)$ is uniformly bounded from below, and the point p is uniformly far from the edges of $L_U(p)$, if any. Hence, there is $\delta > 0$ such that, for every periodic point $p \in \Lambda$, the central curve $L_U(p)$ contains a disk centered at p of length bigger that δ.

As in (12), we classify the periodic points of f in U as follows:

$$P_1 \cup P_2 \cup P_3 \cup P_4 = \{p \in \mathrm{Per}(f) \cap U, \quad \text{period of } p \geq K\}$$

where

- $p \in P_1$ if the extremal points of Γ_p are attracting in the central direction,
- $p \in P_2$ if the extremal points of Γ_p are repelling in the central direction,
- $p \in P_3$ if there are one attracting and one repelling extremal points of Γ_p, and
- $p \in P_4$ if Γ_p is a closed curve.

Remark 6.8 Given a periodic point p in Λ, from the Morse-Smale dynamics induced on the one dimensional curve $L(p)$, there are finitely many periodic points $a_1, \dots, a_{m_p}$ of index s and finitely many periodic points $b_1, \dots, b_{n_p}$ of index $s+1$ such that

$$\Gamma_p \subset \{a_1, \dots, a_{m_p}\} \cup \bigcup_{i=1}^{n_p} W^s(b_i, f) \quad \text{and} \quad \Gamma_p \subset \{b_1, \dots, b_{n_p}\} \cup \bigcup_{i=1}^{m_p} W^u(a_i, f).$$

Remark 6.9 If $p \in P_1 \cup P_4$ then $\partial\Gamma_p$ is either the empty set or consists of (at most two) points of index $s+1$. Hence

$$L_U(p) \subset \Gamma_p \cup W^s(\partial\Gamma_p).$$

Similarly, if $p \in P_2 \cup P_4$, then $\partial\Gamma_p$ is either the empty set or consists of (at most two) points of index s. Hence

$$L_U(p) \subset \Gamma_p \cup W^u(\partial\Gamma_p).$$

In this case, as Λ is an attractor, both Γ_p and $W^u(\partial\Gamma_p)$ are subsets of Λ, and thus $L_U(p) \subset \Lambda$.

Lemma 6.10 *For every $p \in \mathrm{Per}(f) \cap U$ with period bigger than K, the following properties hold:*

1. $\Gamma_p \subset \Lambda_f(U)$,
2. $f(\Gamma_p) = \Gamma_{f(p)}$,
3. $f(P_i) = P_i$.

Proof: By definition, the periodic points of Γ_p belong to U. Also note that

$$\Gamma_p \subset \bigcup_{q \in \Gamma_p} W^u(q)$$

and that, as $\Lambda_f(U)$ is an attractor, $W^u(q) \subset \Lambda_f(U)$, proving item (1).

Item (2) and (3) follow directly from item (1) and the coherent choice of the central curves. Observe that the diffeomorphism f send closed curves to closed curves and the extremal points of Γ_p to the extremal points of $\Gamma_{f(p)}$. ■

Lemma 6.11 $\overline{P_i} = \Lambda_f(U)$ *for some* $i \in \{1, ..., 4\}$.

Proof: By item (2) of Theorem 4.3 the periodic points of f are dense in $\Lambda_f(U)$. By Remark 6.6, the periodic points of $\Lambda_f(U)$ of period less then K (given by Theorem 6.5) is a finite set. Since $\Lambda_f(U)$ is infinite and transitive, it has no isolated periodic orbits, and then $\Lambda_f(U) = \overline{P_1} \cup \overline{P_2} \cup \overline{P_3} \cup \overline{P_4}$. Let $x \in \Lambda_f(U)$ be a point with dense orbit. Then $x \in \overline{P_i}$ for some $i \in \{1, ..., 4\}$. From item (3) in Lemma 6.10, the whole orbit of x lies in $\overline{P_i}$. Consequently $\overline{P_i} = \Lambda_f(U)$. ■

6.2
Proof of Theorem 6.1

Before proving Theorem 6.1, we state some auxiliary lemmas. In what follows we always assume (without mentioning) that $\Lambda_f(U)$ is a $(s, 1, u)$-partially hyperbolic attractor.

Given $n \in \mathbb{N}$, let $\mathrm{Per}(n, f_{|_U})$ be the set of periodic points in $\Lambda_f(U)$ whose period is less than or equal to n. By Remark 6.6, for every $f \in \mathcal{R}$ and $n \in \mathbb{N}$, the set $\mathrm{Per}(n, f_{|_U})$ is a finite (hyperbolic) set. This immediately implies the following remark.

Remark 6.12 Given $f \in \mathcal{R}$ and $n \in \mathbb{N}$, there is a neighborhood $\mathcal{U}_n$ of f such that, for every $g \in \mathcal{U}_n$, the set $\mathrm{Per}(n, g_{|_U})$ consists exactly of the continuations of the points in $\mathrm{Per}(n, f_{|_U})$.

The next lemma follows using a standard Kupka-Smale-like argument.

Lemma 6.13 *Fix $n \in \mathbb{N}$. Let $\mathcal{U}$ be a compatible neighborhood of $f \in \mathcal{R}$ such that, for $g \in \mathcal{U}$, the set $\mathrm{Per}(n, g_{|_U})$ is the continuation of the hyperbolic set $\mathrm{Per}(n, f_{|_U})$. Fixed $\varepsilon > 0$, there is an open and dense subset $\mathcal{V}$ of $\mathcal{U}$ such that, for every $g \in \mathcal{V}$ and every pair of distinct points $p_g, q_g \in \mathrm{Per}(n, g_{|_U})$ it holds that*

$$\mathcal{F}^s_\varepsilon(p_g, g) \cap \mathcal{F}^u_\varepsilon(q_g, g) = \emptyset.$$

Proof: Fix $p, q \in \mathrm{Per}(n, f_{|_U})$ with $p \neq q$. Observe that by the continuous dependence of the leafs on g, if $\mathcal{F}^s_\varepsilon(p_g, g) \cap \mathcal{F}^u_\varepsilon(q_g, g) = \emptyset$, than it holds in an open neighborhood of g.

On the other hand, since $u + s$ is less than the ambient dimension, if $\mathcal{F}^s_\varepsilon(p_g, g) \cap \mathcal{F}^u_\varepsilon(q_g, g) \neq \emptyset$, this intersection is not transverse. Hence, after an arbitrarily small perturbation, we can assume that the disks $\mathcal{F}^s_\varepsilon(p_g, g)$ and $\mathcal{F}^u_\varepsilon(q_g, g)$ are disjoint. As a conclusion, there is an open and dense subset $\mathcal{V}_{p,q}$ of $\mathcal{U}$ such that, for every $g \in \mathcal{V}_{p,q}$ it holds that

$$\mathcal{F}^s_\varepsilon(p_g, g) \cap \mathcal{F}^u_\varepsilon(q_g, g) = \emptyset. \tag{6.2.1}$$

By Remark 6.12, the set

$$\mathcal{V} = \bigcap_{(p,q)\in\mathcal{B}} \mathcal{V}_{p,q}\ , \ \text{where } \mathcal{B} = \{(p, q) \in \{\mathrm{Per}(n, f_{|_U})\}^2 \mid p \neq q\},$$

is a finite intersection of open and dense subsets of $\mathcal{U}$, so $\mathcal{V}$ is an open and dense subset of $\mathcal{U}$. By construction, every $g \in \mathcal{U}$ satisfies Equation (6.2.1). ∎

Next result follows from Lemma 6.13 using a standard genericity argument.

Lemma 6.14 *Let $\mathcal{U}$ be a compatible neighborhood of $f \in \mathcal{R}$ with respect to the attractor $\Lambda_f(U)$. Fixed $\varepsilon > 0$, there is a residual subset $\mathcal{G}$ of $\mathcal{U}$ such that for every $g \in \mathcal{G}$ and every pair of periodic points a, b of $\Lambda_g(U)$ it holds that*

$$\mathcal{F}^s_\varepsilon(a,g) \cap \mathcal{F}^u_\varepsilon(b,g) = \emptyset.$$

Proof: Let $\{f_i\}_{i\in\mathbb{N}}$ be a dense subset of $\mathcal{U}$. Fixed $n \in \mathbb{N}$, we apply Lemma 6.13 to each f_i with respect to n, obtaining an open and dense subset $\mathcal{V}_i^n$ of a neighborhood of f_i satisfying the conclusion of Lemma 6.13. Note that $\mathcal{G}_n = \bigcup_{i\in\mathbb{N}} \mathcal{V}_i$ is an open and dense subset of $\mathcal{U}$. Finally, setting $\mathcal{G} = \bigcap_{n\in\mathbb{N}} \mathcal{G}_n$ we obtain the residual subset of $\mathcal{U}$ satisfying the conclusion in the lemma. ∎

Note that, since Λ is an attractor, for any $x \in \Lambda$ and $\varepsilon > 0$ the disk $\mathcal{F}^u_\varepsilon(x)$ is a subset of Λ. Then, $\mathcal{F}^s_\varepsilon(z)$ is well defined for every $z \in \mathcal{F}^u_\varepsilon(x)$. Given $x \in \Lambda$, consider the topological disk of codimension one

$$\Delta(x,\varepsilon) = \bigcup_{z\in\mathcal{F}^u_\varepsilon(x)} \mathcal{F}^s_\varepsilon(z). \tag{6.2.2}$$

Recall that, for every periodic point p, the central curve $L_U(p)$ is tangent to the bundle E^c, and consequently transverse to $E^s \oplus E^u$. By Remark 6.7, $L_U(p)$ contains a disk centered at p of length δ. Recall that δ does not depend on p. Then, if p is sufficiently close to x, the central curve $L_U(p)$ meets topologically transversely the disk $\Delta(x,\varepsilon)$, say at the point z_p (see Figure 6.1).

Lemma 6.15 *If z_p lies in the stable manifold of some periodic point $\tilde{p} \neq z_p$ in Γ_p, then $\tilde{p} \in \overline{\mathcal{O}_f(\mathcal{F}^u(x))}$.*

Proof:

By the definition of $\Delta(x,\varepsilon)$ and z_p, there is a point $w_p \in \mathcal{F}^u_\varepsilon(x)$ such that $z_p \in \mathcal{F}^s_\varepsilon(w_p)$. Then

$$\lim_{n\to\infty} d(f^n(z_p), f^n(w_p)) = 0. \tag{6.2.3}$$

Since, by hypothesis, $z_p \in W^s(q,f)$, we get that

$$\lim_{n\to\infty} d(f^n(z_p), f^n(\tilde{p})) = 0. \tag{6.2.4}$$

From Equations (6.2.3) and (6.2.4) we obtain that

$$\lim_{n\to\infty} d(f^n(\tilde{p}), f^n(w_p)) \to 0.$$

Since $w_p \in \mathcal{F}^u(x)$, this implies that the orbit of the leaf $\mathcal{F}^u(x)$ accumulates at $\tilde{p}$. ∎

For notational simplicity let us replace f by f_* in the statement of Theorem 6.1 and reserve the symbol f to be a perturbation of f_*. In this way, we adopt the notations of Sections 1, 2, and 8, where f is omitted (if no misunderstanding occurs).

Given $f_* \in \mathcal{R}$, let $\Lambda_{f_*}(U)$ be a $(s,1,u)$-partially hyperbolic proper attractor and $\mathcal{U}_*$ be a compatible neighborhood of f_*. Let $\mathcal{G}_*$ be the residual subset of $\mathcal{U}_*$ given by Lemma 6.14 for f_*.

To prove Theorem 6.1, it suffices to see that for every $f \in \mathcal{G}_* \cap \mathcal{R}$ either $\Lambda_f(U)$ is s-minimal or it is u-minimal.

End of the proof of Theorem 6.1.

Fix $f \in \mathcal{G}_* \cap \mathcal{R}$ and $\Lambda = \Lambda_f(U)$. We split the proof of the theorem into several cases, according to which set P_i (see Section 6.1) is dense in Λ. Theorem 6.1 is an immediate consequence of the following proposition.

Proposition 6.16

1. *If the set P_1 is dense in Λ then Λ is u-minimal.*
2. *If the set P_2 is dense in Λ then Λ is s-minimal.*
3. *If the set P_3 is dense in Λ then Λ is u-minimal.*
4. *If the set P_4 is dense in Λ then Λ is simultaneously s and u-minimal.*

Proof: We consider three cases that imply the proposition.

Case (a): If the set $P_1 \cup P_4$ is dense in Λ, then Λ is u-minimal.

As $f \in \mathcal{R}$, Λ is a homoclinic class (see Remark 5.8). By Theorem 5.12, to prove the u-minimality of Λ it is enough to see that, for every $x \in \Lambda$, it holds that

$$\overline{\mathcal{O}(\mathcal{F}^u(x))} \cap \mathrm{Per}_{s+1}(f_{|\Lambda}) \neq \emptyset. \tag{6.2.5}$$

Since, by hypotheses, the set $P_1 \cup P_4$ is dense in Λ, if $p \in P_1 \cup P_4$ is sufficiently close to x, then as in Lemma 6.15 there is a transverse intersection point z_p between $L_U(p)$ and $\Delta(x,\varepsilon)$ (see Figure 6.1).

By Remarks 6.8 and 6.9, the point z_p is either in a stable manifold of some periodic point $\tilde{p} \in \Gamma_p$ of index $s+1$, or z_p is a hyperbolic periodic point of index s.

Suppose that the first possibility holds. Then Lemma 6.15 implies Equation (6.2.5) to this situation.

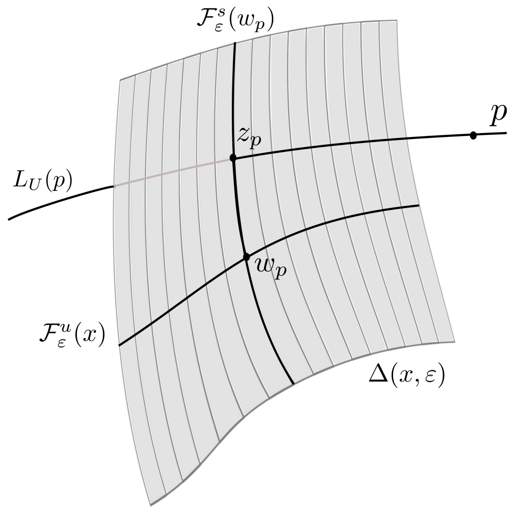

Figure 6.1: Case (a)

Now suppose that z_p is a hyperbolic periodic point of index s. From Equation (6.2.3) it follows that the orbit of $\mathcal{F}^u(x)$ accumulates on the orbit of $\mathcal{F}^u(z_p)$. Thus to get Equation (6.2.5) it is enough to show that

$$\overline{\mathcal{O}(\mathcal{F}^u(z_p))} \cap \mathrm{Per}_{s+1}(f_{|\Lambda}) \neq \emptyset. \tag{6.2.6}$$

Consider the topological manifold $\Delta(z_p,\varepsilon)$. For any $q \in P_1 \cup P_4$ sufficiently close to z_p, the curve $L_U(q)$ meets topologically transversely $\Delta(z_p,\varepsilon)$ at some point z_q. By Lemma 6.14, z_p is the only periodic point in $\Delta(z_p,\varepsilon)$, and by Remark 6.8, z_q belongs to the stable manifold of some periodic point in $L_U(q)$ with index $s+1$. Now we apply Lemma 6.15 to obtain Equation (6.2.6) and, consequently, Equation (6.2.5).

As this holds for every $x \in \Lambda$, we end the proof of Case (a).

Case (b): If the set $P_2 \cup P_4$ is dense in Λ, then Λ is s-minimal.

Note that, since the points in the strong stable disk may not belong to Λ, we cannot argue as in Case (a) by saturating a strong stable disk with strong unstable leaves. To bypass this difficulty, for each point in $p \in P_2 \cup P_4$ we introduce the following topological disk (see Figure 6.2).

$$\nabla(p,\varepsilon) = \bigcup_{y \in L_{\varepsilon,U}(p)} \mathcal{F}^u_\varepsilon(y). \tag{6.2.7}$$

Note that, by Remark 6.9, the curve $L_{\varepsilon,U}(p)$ is contained in Λ. Thus $\mathcal{F}^u_\varepsilon(y)$ is well defined for every small $\varepsilon > 0$, $p \in P_2 \cup P_4$, and $y \in L_{\varepsilon,U}(p)$.

By Theorem 5.12, to prove the s-minimality of Λ it is enough to prove that, for every $x \in \Lambda$, it holds that

$$\overline{\mathcal{O}(\mathcal{F}^s(x))} \cap \mathrm{Per}_s(f_{|\Lambda}) \neq \emptyset. \tag{6.2.8}$$

Recall that, by Remark 6.7, the curve $L_U(p)$ contains a disk centered at p with length δ inside U. Hence, given any $x \in \Lambda$, if $p \in P_2 \cap P_4$ is sufficiently close to x then $\mathcal{F}^s_\varepsilon(x)$ intersects (topologically transversely) $\nabla(p,\varepsilon)$ at some point w_p (see Figure 6.2). By the definition of $\nabla(p,\varepsilon)$, there is $z_p \in L_{\varepsilon,U}(p)$ such that $w_p \in \mathcal{F}^u_\varepsilon(z_p)$, and thus

$$\lim_{n\to\infty} d(f^{-n}(z_p), f^{-n}(w_p)) = 0. \tag{6.2.9}$$

By Remarks 6.8 and 6.9, either $z_p \in W^u(\tilde{p})$ for some periodic point $\tilde{p} \in \Gamma_p \cap \mathrm{Per}_s(f_{|\Lambda})$, or $z_p \in \mathrm{Per}_{s+1}(f_{|\Lambda})$. Similarly to Case (a), in the former situation we get that

$$\lim_{n\to\infty} d(f^{-n}(z_p), f^{-n}(\tilde{p})) = 0. \tag{6.2.10}$$

Combining Equations (6.2.3) and (6.2.4), it follows that

$$\lim_{n\to\infty} d(f^{-n}(\tilde{p}), f^{-n}(w_p)) \to 0,$$

which implies Equation (6.2.8), since $w_p \in \mathcal{F}^s(x)$.

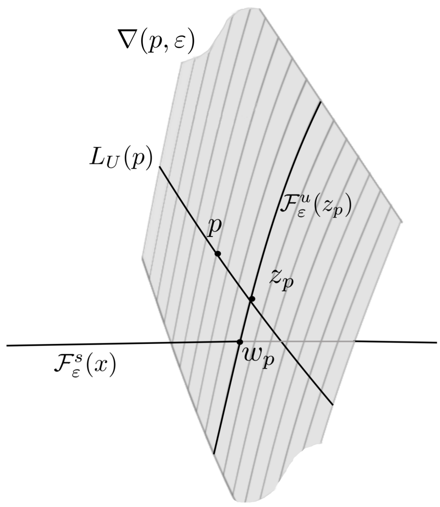

Figure 6.2: Case (b)

In the latter situation, where $z_p \in \mathrm{Per}_{s+1}(f_{|\Lambda})$, the orbit of $\mathcal{F}^s(x)$ accumulates on the orbit of $\mathcal{F}^s(z_p)$. Then, to conclude Equation (6.2.8), it is enough to prove that

$$\overline{\mathcal{O}(\mathcal{F}^s(z_p))} \cap \mathrm{Per}_s(f_{|\Lambda}) \neq \emptyset. \tag{6.2.11}$$

Consider a periodic point $q \in P_2 \cup P_4$ close to z_p and the topological manifold $\nabla(q,\varepsilon)$ such that $\mathcal{F}^s_\varepsilon(z_p)$ intersects topologically transversely $\nabla(q,\varepsilon)$ at a point w_q. Then, there exist a point $z_q \in L_U(q)$ such that $w_q \in \mathcal{F}^u_\varepsilon(z_q)$. By

Lemma 6.14, z_q cannot be a periodic point, so the only possibility is that z_q is in the unstable manifold of some periodic point of Γ_q of index s. Hence we fall in the same situation we have treated in the first part of the proof, obtaining Equation (6.2.11) and, consequently, Equation (6.2.8).

As this holds for every $x \in \Lambda$, we end the proof of Case (b).

Case (c): If the set P_3 is dense in Λ, then Λ is u-minimal.

By Theorem 5.12, to prove the u-minimality of Λ it is enough to prove that, for every $x \in \Lambda$, Equation (6.2.5) holds.

Consider the codimension one topological disk $\Delta(x,\varepsilon)$, see Equation (6.2.2). Fix $\tilde{\varepsilon}$ and $p \in P_3$ sufficiently close to x so that $L_{\tilde{\varepsilon},U}(p)$ intersects topologically transversely $\Delta(x,\varepsilon)$, say at the point z_p. Let l be the curve interval joining z_p and p inside $L_U(p)$. We assume that $\tilde{\varepsilon}$ is sufficiently small so that every periodic point q close to l has its central curve $L_U(q)$ meeting topologically transversely $\Delta(x,\varepsilon)$. We also assume that there is no periodic point in the interior of the curve l. Otherwise, we replace p by a periodic point in $L_U(p)$ with this property. If $z_p = p$ then l is trivial and we fall in a particular case of item (i) in the following list of possibilities.

By Remarks 6.8 and 6.9, there are three possible configurations:

(i) either z_p is a periodic point or

(ii) $z_p \in W^s(p)$ (if p has index $s+1$) or

(iii) $z_p \in W^u(p)$ (if p has index s).

If $z_p \in W^s(p)$, then Equation (6.2.5) follows from Lemma 6.15. If $\mathrm{Per}_{s+1}(\mathrm{f}_{|\Lambda})$ then, as in case (a), Lemma 6.15 gives quation (6.2.6) and, consequently, Equation (6.2.5). This ends item (ii) and half of item (i). The other half of item (1) is when $z_p \in \mathrm{Per}_\mathrm{s}(\mathrm{f}_{|\Lambda})$ that will be treat later.

Let us consider item (iii) where $z_p \in W^u(p)$. Note that in this case the segment $l \subset L_U(p)$ joining p and z_p is a subset of the unstable manifold of p, so it is contained in the attractor Λ.

Let $\tilde{z}_p = f^{-2d}(z_p) \in l$ and $\tilde{\Delta}(x,\varepsilon) = f^{-2d}(\Delta(x,\varepsilon))$, where d is the period of p (see Figure 6.3). Denote by $\tilde{l}$ the curve joining z_p and $\tilde{z}_p$ inside l. Since the curve $\tilde{l}$ is a subset of Λ, it is accumulated by periodic points of Λ.

If q is close to $\tilde{z}_p$ then $L_U(q)$ meets $\tilde{\Delta}(x,\varepsilon)$ transversely. So we take a point $q \in P_3$ such that there are points $z_q, \tilde{z}_q$ given by

$$z_q = L_U(q) \pitchfork \Delta(x,\varepsilon) \quad \text{and} \quad \tilde{z}_q = L_U(q) \pitchfork \tilde{\Delta}(x,\varepsilon), \tag{6.2.12}$$

such that q lies in the curve bounded by z_q and $\tilde{z}_q$ inside $L_U(q)$ (see Figure 6.3).

If the point z_q belongs to the stable manifold of some periodic point of index $s+1$ or z_q is a periodic point of index s then, as in Case (a), Lemma 6.15 implies Equation (6.2.5). Thus we can assume that z_q does not belong to the stable manifold of any point of $L_U(q)$ and is not a periodic point of index s. Hence the only possibility, as $q \in P_3$, is that z_q lies in the unstable manifold of the extremal point of Γ_q of index s.

On the other hand, as $q \in P_3$ we have that either $\tilde{z}_q \in \Gamma_q$ or $\tilde{z}_q$ is in the stable manifold of the extremal point of Γ_q of index $s+1$.

By the coherent choice of the central curves and Equation 6.2.12, the curve $L_U(f^{2d}(q)) = f^{2d}(L_U(q))$ meets transversely $\Delta(x, \varepsilon)$ at the point $f^{2d}(\tilde{z}_q)$. By item (2) of Lemma 6.10, this intersection is either in $\Gamma_{f^{2d}(q)}$ or lies in the stable manifold of the extremal point of $\Gamma_{f^{2d}(q)}$. In any case, we are again in one of the situations treated in Case (a), where we apply Lemma 6.15 to obtain Equation (6.2.5). This ends item (iii).

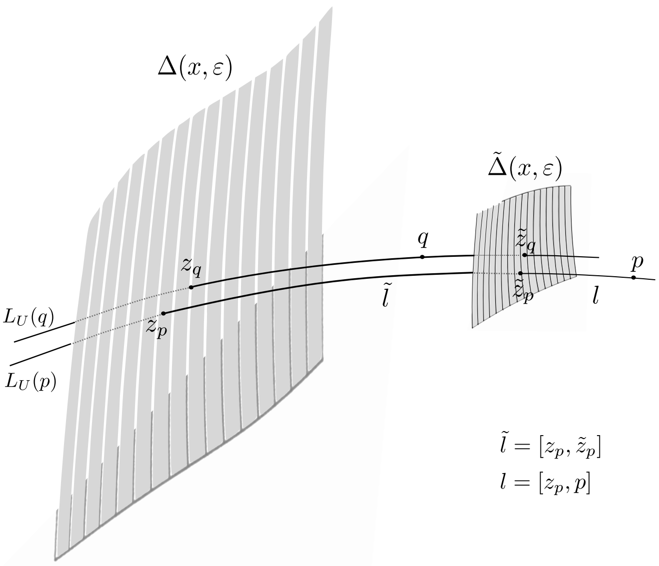

Figure 6.3: Case (c)

We are left with the last possibility in item (i), that is: $z_p \in \mathrm{Per_s}(\mathrm{f}_{|\Lambda})$. Observe that there exist $w_p \in \mathcal{F}^u_\varepsilon(x)$ such that $z_p \in \mathcal{F}^s_\varepsilon(w_p)$. This implies that the orbit of $\mathcal{F}^s(x)$ accumulates on the orbit of $\mathcal{F}^s(z_p)$. So to conclude Equation (6.2.8) it is enough to prove that

$$\overline{\mathcal{O}(\mathcal{F}^u(z_p))} \cap \mathrm{Per}_s(f_{|\Lambda}) \neq \emptyset. \tag{6.2.13}$$

To obtain this equation, we follow all the initial arguments in this proof replacing the point x with z_p.

Consider a point $q \in P_3$ sufficiently close to z_p so that $L_{\tilde{\varepsilon},U}(q)$ intersects topologically transversely $\Delta(z_p, \varepsilon)$ in a point z_q. In principle, we should verify all the three items (i), (ii), and (iii) to this new configuration. However, observe that items (ii) and (iii) was already verified in the general case (for every $x \in \Lambda$) in the scope of this proof. In addition, by Lemma 6.14, z_p is the only periodic point in $\Delta(z_p, \varepsilon)$, so item (i) do not occur. Hence we obtain Equation (6.2.13) and, consequently, Equation (6.2.5). This ends Case (c).

This completes the analysis of the three cases we need to consider to prove Proposition 6.16. ■