2
Sea Surface Elevation

Pérez [29] states that ocean waves are random in terms of both time and space It is assumed that the variations of the stochastic characteristics of the sea are much slower than the variations of the sea surface itself. Therefore the elevation of the sea at a position x, y, given by $\zeta(x, y, t)$, can be considered a realization of a stationary process. The following simplifying assumptions about the underlying model are usually made:

- The observed sea surface, at a certain location and for short periods of time, is considered a realization of a stationary and homogeneous, zero mean Gaussian stochastic process;
- Under a Gaussian assumption, the process, in a statistical sense, is completely characterized by the power spectral density function S.

The validity of these assumptions have been investigated via analysis of time series recorded from wave riding buoys in the North Atlantic Ocean and it has been reported that

- For low and moderate sea states, significant wave height ($h_{1/3}$) lower than 4 m, the sea can be considered stationary for periods over 20 min. For more severe sea states, stationarity can be questioned even for periods of 20 min.
- For low to medium states, $h_{1/3} < 8\ m$, Gaussian models are still accurate but deviations from Gaussianity slightly increase with the increasing severity of the sea state.

Langley, [21], states that the statistical distribution of the parameters of the sea surface elevation at a particular location is normally determined by measuring values at three hourly intervals over an extended period. In this work it was considered that the parameters of sea surface elevation remain unchanged for three hours.

In this chapter the steps for obtaining the sea surface elevation will be explained.

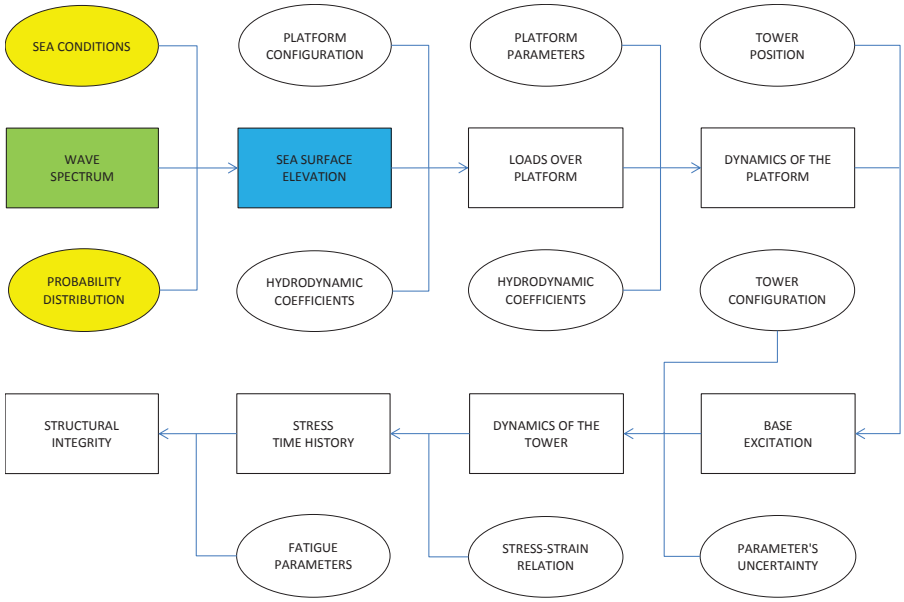

Figure 2.1: Obtaining the sea surface elevation

2.1 Regular Waves

A conceptual model to describe the sea surface elevation is given by the sum of a large number of essentially independent regular (sinusoidal) contributions with random phases. In this representation, the sea surface elevation at a location x, y with respect to a X, Y, and Z global coordinate system is given by [29]

$$\zeta(x,y,t) = \sum_{i=1}^{N} \zeta_i(x,y,t) = \sum_{i=1}^{N} \bar{\zeta}_i \cos\left(k_i x \cos\chi + k_i y \sin\chi + \omega_i t + \theta_i\right) \qquad (2.1)$$

where $\zeta_i(x,y,t)$ is the contribution of the regular or harmonic traveling wave components i progressing at an angle χ with respect to the X direction and with a random phase θ_i. The parameters k_i (wave number), ω_i (wave frequency seen from a fixed position) and $\bar{\zeta}_i$ (constant wave amplitude) characterize each component. For each realization, the phase angle θ_i of each component is chosen to be a random variable with uniform distribution on the interval $[-\pi, \pi]$. This choice ensures the stationarity of $\zeta_i(x,y,t)$ [29].

For each regular wave component i, the velocity with which the wave crest moves relative to the ground, the phase velocity, is given by [29]

$$c_i = \sqrt{\frac{g\lambda_i}{2\pi}}, \qquad i = 1, 2, \ldots, N \qquad (2.2)$$

where g is the gravity acceleration and λ_i is the wavelength of the component i. The wave number is given by [29]

$$k_i = \frac{2\pi}{\lambda_i}, \quad i = 1, 2, \ldots, N \tag{2.3}$$

and the wave frequency is given by [29]

$$\omega_i = \sqrt{gk_i} = \frac{g}{c_i}, \quad i = 1, 2, \ldots, N \tag{2.4}$$

The Eq. (2.4) is known as the dispersion of gravity waves and establishes that the phase velocity is inversely proportional to its frequency. This means that long waves propagate faster than short ones. Considering that the observations are made at the origin of the reference frame and that the waves come from an angle of incidence $\chi = 0$ with respect to the reference frame. In this case, the Eq. (2.1) becomes [29]

$$\zeta(t) = \sum_{i=1}^{N} \zeta_i(t) = \sum_{i=1}^{N} \bar{\zeta}_i \cos\left(\omega_i t + \theta_i\right) \tag{2.5}$$

2.2 Irregular Waves

Since observed waves are not regular, the wave height and frequency are not easily defined. Therefore, the wave height spectral density is utilized for a statistical description of the sea surface elevation. The sea surface elevation can be related to its Fourier transform by [4]

$$\zeta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(-i\omega t) \mathrm{d}\omega \tag{2.6}$$

Considering the sea surface elevation an ergodic process, its mean-square value can be approximated by the time average over a long period of time [4]

$$E\left\{\zeta^2(t)\right\} = \lim_{T_s \to \infty} \frac{1}{T_s} \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \mathrm{d}\omega \tag{2.7}$$

The power spectral density (spectrum) is defined as [4]

$$S_{\zeta\zeta}(\omega) = \frac{1}{2\pi T_s} |X(\omega)|^2 \tag{2.8}$$

and the mean-square of sea surface elevation is given by [4]

$$E\left\{\zeta^2(t)\right\} = \int_{-\infty}^{\infty} S_{\zeta\zeta}(\omega) \mathrm{d}\omega \tag{2.9}$$

The spectrum is related to the autocorrelation function by the Wiener-Khinchine relations [4]

$$S_{\zeta\zeta}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{\zeta\zeta}(\tau)\exp(-i\omega\tau)\mathrm{d}\tau \tag{2.10}$$

$$R_{\zeta\zeta}(\tau) = \int_{-\infty}^{\infty} S_{\zeta\zeta}(\omega)\exp(i\omega\tau)d\omega \tag{2.11}$$

For a zero-mean process, the mean-square value equals the variance. At any particular wave frequency ω_i the variance of that component within a band Δ_ω centered at ω_i is approximated by [29]

$$\mathrm{var}\left[\zeta_i(t)\right] = \frac{1}{2}\bar{\zeta_i}^2 \approx \int_{\omega_i-\frac{\Delta_\omega}{2}}^{\omega_i+\frac{\Delta_\omega}{2}} S_{\zeta\zeta}(\omega)d\omega \tag{2.12}$$

and the amplitudes of the wave components can be approximated by [29]

$$\bar{\zeta_i} \approx \sqrt{2\int_{\omega_i-\frac{\Delta_\omega}{2}}^{\omega_i+\frac{\Delta_\omega}{2}} S_{\zeta\zeta}(\omega)d\omega} \tag{2.13}$$

For ocean applications, a one-sided spectrum given in Hertz (Hz) is often used. For this one-sided spectrum, a superscript o is given and it can be obtained from the two-sided spectrum by the relation [4]

$$S^o_{\zeta\zeta}(\omega) = 2S_{\zeta\zeta}(\omega), \quad \omega \geq 0 \tag{2.14}$$

The two-sided spectrum given in radians can be transformed to the spectrum given in Hertz by the relation

$$S_{\zeta\zeta}(f) = 2\pi S_{\zeta\zeta}(\omega) \tag{2.15}$$

and the two-sided spectrum given in radians can be transformed to the one-sided spectrum given in Hertz by the relation

$$S^o_{\zeta\zeta}(f) = 4\pi S_{\zeta\zeta}(\omega), \quad f, \omega > 0 \tag{2.16}$$

2.3 Short-term Statistics

An irregular sea state is described by one of its statistics named significant wave height. This statistic is the average height of the highest one-third of all waves and it is found that the observed wave height is consistently very close to the significant wave height [4].

When describing short-term statistics two assumptions are made, namely, stationarity and ergodicity. These assumptions are valid only for short time intervals. The wave elevation is assumed to be weakly stationary so that its autocorrelation is a function of time lag only. As a result, the mean and the variance are constant and the spectral density is invariant with time and the

significant wave height and the significant wave period are constant when considering short term statistics. In this case the individual wave height and wave period are the random variables.

The rate at which the random process ζ crosses an elevation represented by the random variable Z with a positive slope is given by [4]

$$\nu_{z^+} = \int_0^\infty v f_{\zeta\dot{\zeta}}(z, v)\mathrm{d}v \tag{2.17}$$

where $f_{\zeta\dot{\zeta}}$ is a joint probability density function. The expected time of the first up-crossing is given by [4]

$$E\{T\} = 1/\nu_{z^+} \tag{2.18}$$

The probability density function of the maxima is given by [4]

$$f_A(a) = \frac{\int_{-\infty}^0 -w f_{\zeta\dot{\zeta}\ddot{\zeta}}(a, 0, w)\mathrm{d}w}{\int_{-\infty}^0 w f_{\dot{\zeta}\ddot{\zeta}}(0, w)\mathrm{d}w} \tag{2.19}$$

where $f_{\zeta\dot{\zeta}\ddot{\zeta}}$ is a joint probability density function. If ζ is a Gaussian process the joint probability density functions are [4]

$$f_{\zeta\dot{\zeta}}(x, \dot{x}) = \frac{1}{2\pi\sigma_\zeta\sigma_{\dot{\zeta}}}\exp\left[-\frac{1}{2}\left(\frac{x}{\sigma_\zeta}\right)^2 - \frac{1}{2}\left(\frac{\dot{x}}{\sigma_{\dot{\zeta}}}\right)^2\right],$$

$$-\infty < x < \infty, -\infty < \dot{x} < \infty \tag{2.20}$$

and [4]

$$f_{\zeta\dot{\zeta}\ddot{\zeta}}(x, \dot{x}, \ddot{x}) = \frac{1}{(2\pi)^{3/2}|M|^{1/2}}\exp\left[-\frac{1}{2}\left(\{x\} - \{\mu_\zeta\}\right)^T [M]^{-1}\left(\{x\} - \{\mu_\zeta\}\right)\right] \tag{2.21}$$

where

$$[M] = \begin{bmatrix} \sigma_\zeta^2 & 0 & \sigma_{\dot{\zeta}}^2 \\ 0 & \sigma_{\dot{\zeta}}^2 & 0 \\ \sigma_{\dot{\zeta}}^2 & 0 & \sigma_{\ddot{\zeta}}^2 \end{bmatrix} \tag{2.22}$$

and

$$x - \mu_\zeta = \begin{bmatrix} x - \mu_\zeta \\ \dot{x} - \mu_{\dot{\zeta}} \\ \ddot{x} - \mu_{\ddot{\zeta}} \end{bmatrix} \tag{2.23}$$

Then, for a stationary Gaussian process, the up-crossing rate is given by [4]

$$\begin{aligned}\nu_z^+ &= \int_0^\infty f_{\zeta\dot{\zeta}}(Z,\dot{x})\dot{x}\mathrm{d}\dot{x} \\ &= \frac{1}{2\pi\sigma_\zeta\sigma_{\dot{\zeta}}}\exp\left[-\frac{1}{2}\left(\frac{Z}{\sigma_\zeta}\right)^2\right]\int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{\dot{x}}{\sigma_{\dot{\zeta}}}\right)^2\right]\dot{x}\mathrm{d}\dot{x} \\ &= \frac{\sigma_{\dot{\zeta}}}{2\pi\sigma_\zeta}\exp\left[-\frac{1}{2}\left(\frac{Z}{\sigma_\zeta}\right)^2\right]\end{aligned} \tag{2.24}$$

and the probability density function of maxima is given by the Rice density function

$$\begin{aligned}f_A(a) &= \frac{\sqrt{1-\alpha^2}}{\sqrt{2\pi}\sigma_\zeta}\exp\left(\frac{-a^2}{2\sigma_\zeta^2\left(1-\alpha^2\right)}\right) \\ &+a\frac{\alpha}{\sigma_\zeta^2}\Phi\left(\frac{a\alpha}{\sigma_\zeta\sqrt{\left(\alpha^2-1\right)}}\right)\exp\left(\frac{-a^2}{2\sigma_\zeta^2}\right)\end{aligned} \tag{2.25}$$

where Φ is the cumulative distribution function of the standard normal random variable given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\,\exp\left(-z^2/2\right)\mathrm{d}z \tag{2.26}$$

and α is the irregularity factor, equivalent to the ratio of the number of zero up-crossings to the number of peaks. This factor ranges from 0 to 1 and it is also equal to

$$\alpha = \frac{\sigma_{\dot{\zeta}}^2}{\sigma_\zeta\sigma_{\ddot{\zeta}}} \tag{2.27}$$

If ζ is a broad-band process, then $\alpha = 0$ and the Rice distribution is reduced to the Gaussian probability density function given by

$$f_G(a) = \frac{1}{\sqrt{2\pi}\sigma_\zeta}\exp\left(\frac{-a^2}{2\sigma_\zeta^2}\right) \quad \text{for } -\infty < a < \infty \tag{2.28}$$

If ζ is a narrow-band process, it is guaranteed that it will have a peak whenever it crosses its mean. In this case the irregularity factor is close to unity and the Rice distribution is reduced to the Rayleigh probability density function given by

$$f_R(a) = \frac{a}{\sigma_\zeta^2}\exp\left(\frac{-a^2}{2\sigma_\zeta^2}\right) \quad \text{for } 0 < a < \infty \tag{2.29}$$

That means that the amplitudes of a narrow-band stationary Gaussian process are distributed according to the Rayleigh distribution.

The maxima of ζ, A, are the amplitudes of the sea surface elevation. The

wave height, $H = 2A$, is then distributed according to [4]

$$\begin{aligned} f_H(h) &= f_R(H/2)\frac{\mathrm{d}A}{\mathrm{d}H} \\ &= \frac{h}{4\sigma_\zeta^2}\exp\left(-\frac{1}{2}\frac{h^2}{4\sigma_\zeta^2}\right) \quad \text{for } 0 < h < \infty \end{aligned} \tag{2.30}$$

For any given wave the probability that the height is less than h (the cumulative distribution) is

$$F_H(h) = 1 - \exp\left(-\frac{1}{2}\frac{h^2}{4\sigma_\zeta^2}\right) \quad \text{for } 0 < h < \infty \tag{2.31}$$

If ζ is a stationary narrow-band process so that the peaks are distributed according to the Rayleigh distribution the root mean square of wave height is given by

$$\sqrt{E\{H^2\}} = \int_0^\infty h^2 f_H(h)\mathrm{d}h = 2\sqrt{2}\sigma\zeta \tag{2.32}$$

In addition, it can be shown that the average wave height is given by

$$H_O \equiv E\{H\} = \sqrt{2\pi}\sigma_\zeta \tag{2.33}$$

and the significant wave heights is given by

$$H_S \equiv E\{H_{1/3}\} = 4\sigma_\zeta \tag{2.34}$$

where $E\{H_{1/3}\}$ is the expectation of the highest one-third of the waves.

2.4 Wave Spectrum

In any particular sea state, the sea surface elevation presents irregular characteristics. After the wind has blown constantly for a certain period of time the sea surface elevation becomes stationary. In this case the sea is referred to as *fully-developed*. If the irregularity of the observed waves is only in the dominant wind direction so that there are mainly uni-dimensional wave crests with carrying separation and remaining parallel to each other, the sea is referred to as a *long-crested* irregular sea, [29]. For a fully-developed sea the Pierson-Moskowits (PM) spectrum for the wave amplitudes in terms of the wind velocity is given by [4]

$$S_{\zeta\zeta}^o(\omega) = \frac{8.1\times10^{-3}g^2}{\omega^5}\exp\left(-0.74\left(\frac{g}{V_w}\right)^4\omega^{-4}\right) \tag{2.35}$$

where g is the gravitational constant and V_w is the wind speed at a height of 19.5m above the still water. The PM spectrum can be written in terms of the modal frequency, ω_m, the one at which the spectrum is maximum. In this case it is given by

$$S^o_{\zeta\zeta}(\omega) = \frac{8.1 \times 10^{-3} g^2}{\omega^5} \exp\left(-1.25\omega_m^4 \omega^{-4}\right) \tag{2.36}$$

In some cases it may be necessary to express the spectrum in terms of the significant wave height. For a narrow band Gaussian process the significant wave height is related to the standard deviation of the sea surface elevation by Eq. (2.34), then the spectrum is given by [4]

$$S^o_{\zeta\zeta}(\omega) = \frac{8.1 \times 10^{-3} g^2}{\omega^5} \exp\left(-0.0324 \left(\frac{g}{H_S}\right)^2 \omega^{-4}\right) \tag{2.37}$$

The PM spectrum is applicable for deep water, unidirectional seas, fully developed and local-wind generated with unlimited fetch and was developed for the North Atlantic. The effect of swell is not accounted for in this spectrum and it is found that even though it was derived for the North Atlantic the spectrum is valid for other locations [4].

2.5 Long-term Statistics

In order to predict the possible sea surface elevations that the offshore platform can be subjected to it is necessary to know the values of significant wave heights and its probability of occurrence for the location where it will be installed. For long-term statistics the significant wave height follows the Weibull distribution closely. The probability density function for a three parameter Weibull distribution is given by [4]

$$f_W(H_S) = \frac{m}{\beta}\left(\frac{H_s - \gamma}{\beta}\right)^{m-1} \exp\left(-\left(\frac{H_S - \gamma}{\beta}\right)^m\right) \quad \gamma < H_S \tag{2.38}$$

and the probability distribution function is given by

$$F_W(H_S) = 1 - \exp\left(-\left(\frac{H_S - \gamma}{\beta}\right)^m\right) \quad \gamma < H_S \tag{2.39}$$

where γ, β and m are the Weibull parameters that can be determined by least-squares methods, provided that significant wave height data over a long period of time are available.

The National Data Buoy Center (NBDC) provides historical data about significant wave height collected from several stations all over the world. It maintains a network of data collecting buoys and coastal stations. The buoys

measure wave height and period, and sea surface temperature as well.

Most of the commonly used probability density functions can be obtained from Weibull's equation by the proper choice of the parameters in that equation [27]. The Rayleigh probability density function for the significant wave height is given by [27]

$$f_R(H_S) = \frac{2H_S}{H_{rms}^2}\exp\left(-\left(\frac{H_S}{H_{rms}}\right)^2\right) \tag{2.40}$$

and by comparing the expressions in Eqs. (2.38) and (2.40) it can be noted that the Rayleigh probability density function corresponds to the parametric values of $m = 2$, $\gamma = 0$ and $\beta = H_{rms}$. The mean wave height is given by

$$H_{avg} = \int_0^\infty H_S f(H_S)\mathrm{d}H_S \tag{2.41}$$

where f is a probability density function and considering the Weibull distribution for the significant wave height the mean value is given by

$$H_{avg} = \beta\Gamma\left(\frac{m+1}{m}\right) + \gamma \tag{2.42}$$

where Γ is the Gamma function given by

$$\Gamma(\frac{m+1}{m}) = \int_0^\infty H_S^{\frac{m+1}{m}-1}\mathrm{e}^{-H_S}\mathrm{d}H_S = \int_0^\infty H_S^{1/m}\mathrm{e}^{-H_S}\mathrm{d}H_S \tag{2.43}$$

The mean square significant wave height is given by

$$\bar{H}^2 = \int_0^\infty H_S^2 f(H_S)\mathrm{d}H_S \tag{2.44}$$

and considering the Weibull distribution for the significant wave height the mean square and the root-mean-square are given by

$$\bar{H}^2 = H_{rms}^2 = \beta^2\Gamma\left(\frac{m+2}{m}\right) + 2\gamma\beta\Gamma\left(\frac{m+1}{m}\right) + \gamma^2 \tag{2.45}$$

Etube [9] states that for locations at North Sea the probability distribution function of significant wave height can be given by the Gumbel distribution

$$F_G(H_s) = \eta\exp\left(-\exp\left(\frac{\alpha - H_s}{\lambda}\right)\right) \tag{2.46}$$

and the probability density function is given by

$$f_G(H_s) = \frac{\eta}{\lambda}\exp\left(\frac{\alpha - H_s}{\lambda}\right)\exp\left(-\exp\left(\frac{\alpha - H_s}{\lambda}\right)\right) \tag{2.47}$$

where α, λ and η are site-dependent parameters.

2.6 Reduced-order Model

The use of a sea state representation with a large number of uncorrelated sources of uncertainty in nonlinear wave body interactions leads to a computational task which may become prohibitively expensive when the statistics of extreme loads and responses are necessary [37]. Given the power spectral density of the signal an optimal set of orthogonal functions, a basis, exists that fits the signal with the minimum number of uncorrelated sources of uncertainty [37]. This basis follows from the spectral decomposition theorems of Loève (1945) and Karhunen (1947) [2]. The Karhunen-Loève (KL) is an optimal basis to construct a reduced order model of the sea surface elevation in the sense that the projection on to the subspace generated by this basis contains the maximal amount of energy for a given number of trial functions [23]. An application for reduced-order models can be found on [30].

The use of KL basis to represent a stochastic process is based on two assumptions: the process is stationary in time and ergodic. For long periods of time the sea surface elevation is not a stationary process, as the statistical distribution of H_s is normally determined by measuring the value of H_S at three hourly interval over an extended period, [21], the process can be considered stationary only for a three hour period.

Considering the sea surface elevation at the coordinates $X = Y = 0$, the autocorrelation function of the signal is given by [37]

$$R(\tau) = E[\zeta(t)\zeta(t+\tau)] = R(-\tau) \tag{2.48}$$

Since the two-sided power spectral density of the signal $\zeta(t)$ is the Fourier transform of the autocorrelation function, Eqs. (2.10) and (2.11), the following standard relation holds [37]

$$\sigma_\zeta^2 = R(0) \tag{2.49}$$

Considering a signal over a finite time interval $(-T, T)$ the Karhunen-Loève theorem states that [37]

$$\zeta(t) = \sum_{n=0}^{\infty} \alpha_n f_n(t) \quad \text{for } -T < t < T \tag{2.50}$$

Since ζ is a stochastic process, the coefficients α_n are independent random variables such that [37]

$$E\left(\alpha_n^2\right) = \kappa_n \tag{2.51}$$

and

$$E\left(\alpha_m \alpha_n\right) = 0 \quad \text{for } m \neq n \tag{2.52}$$

The deterministic functions f_n are solutions of an eigenvalue problem cast in the form of an integral equation of the first kind with the autocorrelation

function as its kernel [37]

$$\int_{-T}^{T} R(t-\tau) f_n(\tau) d\tau = \kappa_n f_n(t) \quad \text{for } n = 0, 1, \ldots \tag{2.53}$$

$$\int_{-T}^{T} f_m(\tau) f_n(\tau) d\tau = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{2.54}$$

$$R(t-\tau) = \sum_{n=0}^{\infty} \kappa_n f_n(t) f_n(\tau) \tag{2.55}$$

$$R(t) = \sum_{n=0}^{\infty} \kappa_n f_n(0) f_n(t) \tag{2.56}$$

$$\sigma_\zeta^2 = R(0) = \sum_{n=0}^{\infty} \kappa_n f_n(0) f_n(0)^2 \tag{2.57}$$

$$\alpha_n = \int_{-T}^{T} \zeta(t) f_n(t) dt \tag{2.58}$$

It can be observed that [37] [2]

- The independent random variables α_n are Gaussian if the signal ζ is Gaussian, which is often the case with ocean waves
- The eigenfunctions f_n are even and odd functions for positive and negative values of their argument in the range $(-T, T)$
- The rate of decay of the eigenvalues κ_n with increasing n suggests the number of the terms that are sufficient to keep in the stochastic series expansion, Eq. (2.50). If this number is small, the signal is governed by a small number of independent sources of uncertainty with statistical properties given by Eqs. (2.51) to (2.58)
- The basis f_n is optimal in the sense that it allows the representation of the autocorrelation function with the minimum number of therms in the series in Eq. (2.55)
- The KL representation maximizes the Shannon entropy measure which reveals the minimum number of terms that are sufficient for the representation of the variability of the signal.

Ritto et al [31] proposed two methods of executing the proper orthogonal decomposition (POD) of the dynamics of the system. In the following sections these two methods will be explained.

2.7 Direct Method

Let the sea surface elevation $\zeta(\cdot, t)$ be a vector field in $\Omega \subset \mathbb{R}^2$ and $t \in \mathbb{R}$, i.e., $\zeta(x, y, t)$. If ζ is decomposed in two parts being one invariant in time, $E[\zeta(\cdot, t)]$, and the second part given by

$$\nu(\cdot, t) = \zeta(\cdot, t) - E[\zeta(\cdot, t)] \tag{2.59}$$

then ν is a stochastic process with zero mean and, as a consequence, its correlation tensor equals its autocorrelation tensor [31]. If ν is real then the spatial autocorrelation function of two of its points is defined by the tensorial product

$$R(\mathbf{x}, \mathbf{x}') = E[\nu(\mathbf{x}, t) \otimes \nu(\mathbf{x}', t)] \tag{2.60}$$

Using the ergodicity hypothesis, one can write

$$R(\mathbf{x}, \mathbf{x}') = \frac{1}{\tau} \int_0^\tau \nu(\mathbf{x}, t)\nu(\mathbf{x}', t)\mathrm{d}t \tag{2.61}$$

where τ is the duration of the analysis. The eigenvalues (or proper orthogonal values, POVs) and the eigenfunctions (or proper orthogonal modes, POMs) are computed solving the following eigenvalue problem

$$\int_\Omega R(\mathbf{x}, \mathbf{x}')\, \psi_k(\mathbf{x})\mathrm{d}\mathbf{x} = \lambda_k \psi_k(\mathbf{x}) \tag{2.62}$$

Considering the discretized field

$$\zeta(x_i, y_j, t) \tag{2.63}$$

where i, j assume values from 1 to N_x, N_y respectively. For each instant of time there are N sample values, $N = 2 \times N_x \times N_y$. The number 2 multiplying the expression is due to the two dimensional fields, ζ_x and ζ_y.The sample can be ordered: $\zeta(\mathbf{x}_1, \cdot), \zeta(\mathbf{x}_2, \cdot), \ldots, \zeta(\mathbf{x}_N, \cdot)$. The dynamic system displacement are numerically calculated in N points in M instants of time

$$[Z] = [\zeta(\mathbf{x}_1, \cdot), \ldots, \zeta(\mathbf{x}_N, \cdot)] = \begin{bmatrix} \zeta(\mathbf{x}_1, t_1) & \ldots & \zeta(\mathbf{x}_N, t_1) \\ . & . & . \\ . & . & . \\ . & . & . \\ \zeta(\mathbf{x}_1, t_M) & \ldots & \zeta(\mathbf{x}_N, t_N) \end{bmatrix} \tag{2.64}$$

Using the stationarity and ergodicity assumption, the variation of the field with respect to the mean value is given by

$$[V] = [Z] - \frac{1}{M}\begin{bmatrix} \sum_{i=1}^{M}\zeta\left(\mathbf{x}_1, t_i\right)) & \dots & \sum_{i=1}^{M}\zeta\left(\mathbf{x}_N, t_i\right)) \\ . & . & . \\ . & . & . \\ . & . & . \\ \sum_{i=1}^{M}\zeta\left(\mathbf{x}_1, t_i\right)) & \dots & \sum_{i=1}^{M}\zeta\left(\mathbf{x}_N, t_i\right)) \end{bmatrix} \tag{2.65}$$

and the spatial autocorrelation matrix is given by

$$[R] = \frac{1}{M}[V]^T[V] \tag{2.66}$$

where the matrix $[R]$ is symmetric by construction. The discretized eigenvalue problem is given by

$$[R]\psi_k = \lambda_k\psi_k \tag{2.67}$$

which is the discretized version of Eq. (2.62). The eigenvectors ψ_k (POMs) are used to construct the basis for projection of the dynamics of the system, they are also called Empirical Modes. The POVs are given by the eigenvalues λ_k of the matrix $[R]$. It can be noted that the dimension of matrix $[R]$ depends only on the spatial discretization. Therefore, the use of the direct method is recommended when the spatial mesh is coarse and there are many instants of time.

2.8 Snapshots Method

A snapshot is a configuration of the system at an instant of time. In this method the POMS are computed without using the matrix $[R]$, Eq. (2.66). Substituting the Eq. (2.61) into (2.62) it is obtained

$$\int_\Omega \frac{1}{\tau}\int_0^\tau \nu(\mathbf{x},t)\nu(\mathbf{x}',t)\mathrm{d}t\psi_k(\mathbf{x}')\mathrm{d}\mathbf{x} = \lambda_k\psi_k(\mathbf{x}) \tag{2.68}$$

which can be rewritten as

$$\frac{1}{\tau}\int_0^\tau \nu(\mathbf{x},t)\int_\Omega \nu(\mathbf{x}',t)\psi_k(\mathbf{x}')\mathrm{d}\mathbf{x}\mathrm{d}t = \lambda_k\psi_k(\mathbf{x}) \tag{2.69}$$

and ψ_k can be obtained as

$$\psi_k(\mathbf{x}) = \int_0^\tau \nu(\mathbf{x},t)A_k(t)\mathrm{d}t \tag{2.70}$$

where

$$A_k(t) = \frac{1}{\tau\lambda_k}\int_\Omega \nu(\mathbf{x}',t)\psi_k(\mathbf{x}')\mathrm{d}\mathbf{x} \tag{2.71}$$

which means that $\psi_k(\mathbf{x})$ is a linear combination of $\nu(\mathbf{x},t)$. For a finite number of instants $t_m(m = 1, 2, \dots, M)$, a snapshot is defined as where

$$\nu^{(m)} = \nu\left(\cdot, t_m\right) \tag{2.72}$$

Substituting the Eq. (2.71) in to 2.69 it is obtained

$$\frac{1}{\tau}\int_0^\tau \nu(\mathbf{x},t)\int_\Omega \nu(\mathbf{x}',t)\int_0^\tau \nu(\mathbf{x},t)A_k(t)\mathrm{d}t\mathrm{d}\mathbf{x}\mathrm{d}t = \lambda_k\int_0^\tau \nu(\mathbf{x},t)A_k(t)\mathrm{d}t \quad (2.73)$$

which can be rewritten as

$$\int_0^\tau\int_0^\tau A_k\left(t'\right)D\left(t,t'\right)\mathrm{d}t\mathrm{d}t \quad (2.74)$$

where

$$D\left(t,t'\right) = \frac{1}{\tau}\int_\Omega \nu\left(\mathbf{x}',t\right)\nu\left(\mathbf{x}',t'\right)\mathrm{d}\mathbf{x} \quad (2.75)$$

Discretizing the Eq. (2.74) it is obtained

$$[D]\mathbf{A}_k = \lambda_k\mathbf{A}_k \quad (2.76)$$

The matrix $[D]$ is computed using Eq. (2.65) and 2.75

$$[D] = \frac{1}{M} \quad (2.77)$$

and it has dimensions $M \times M$. The eigenvalues of $[D]$ are the POVs and the POMs are calculated as

$$\psi_k = [V]^T\mathbf{A}_k \quad (2.78)$$

which is the discretized version of Eq. (2.71) and $\mathbf{A}_k$ are the eigenvectors of matrix $[D]$. Therefore, the POMs, ψ_k, are linear combinations of the snapshots, which in turn are the lines of matrix $[V]$.

It can be noted that the dimensions of matrix [D] depend only on the number of snapshots. Therefore, the use of this method is recommended when the spatial mesh is very refined and there are not many instants of time, as in rapidly decaying processes.