

**Marcela Quispe Cruz**

## **Some Results in a Proof-theory Based on Graphs**

**Tese de Doutorado**

Thesis presented to the Programa de Pós Graduação em  
Informática of the Departamento de Informática, PUC-Rio as  
partial fulfillment of the requirements for the degree of Doutor  
em Informática

Advisor : Prof. Edward Hermann Haeusler  
Co-Advisor: Prof. Lew Gordeev

Rio de Janeiro  
November 2014

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## Abstract

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Traditional proof theory of Propositional Logic deals with proofs which size can be huge. Proof theoretical studies discovered exponential gaps between normal or cut free proofs and their respective non-normal proofs. Thus, the use of proof-graphs, instead of trees or lists, for representing proofs is getting popular among proof-theoreticians. Proof-graphs serve as a way to provide a better symmetry to the semantics of proofs and a way to study complexity of propositional proofs and to provide more efficient theorem provers, concerning size of propositional proofs. The aim of this work is to reduce the weight/size of deductions. We present formalisms of proof-graphs that are intended to capture the logical structure of a deduction and a way to facilitate the visualization. The advantage of these formalisms is that formulas and sub-deductions in Natural Deduction, preserved in the graph structure, can be shared deleting unnecessary sub-deductions resulting in the reduced proof. In this work, we give a precise definition of proof-graphs for purely implicational logic, then we extend this result to full propositional logic and show how to reduce (eliminating maximal formulas) these representations such that a normalization theorem can be proved by counting the number of maximal formulas in the original derivation. The strong normalization will be a direct consequence of such normalization, since that any reduction decreases the corresponding measures of derivation complexity. Continuing with our aim of studying the complexity of proofs, the current approach also give graph representations for first order logic, deep inference and bi-intuitionistic logic.

## Keywords

Proof Theory; Proof-graphs; Natural Deduction; Proof Complexity; Strong Normalization; Classical Logic; Implicational Minimal Logic; Deep Inference; Bi-intuitionistic Logic;

## Resumo

Quispe, Marcela; Haeusler, Edward Hermann (orientador); Gordeev, Lew (co-orientador). **Alguns resultados em teoria de prova baseado em grafos**. Rio de Janeiro, 2014. 90p. Tese de Doutorado — Departamento de Informática, Pontifícia Universidade Católica do Rio de Janeiro.

A teoria da prova tradicional da lógica proposicional trata provas cujos tamanhos podem ser demasiado grandes. Estudos teóricos de prova descobriram diferenças exponenciais entre provas normais ou livres de corte e suas respectivas provas não-normais. Assim, o uso de grafos-de-prova, ao invés de árvores ou listas, para representar provas está se tornando mais popular entre teóricos da prova. Os grafos-de-prova servem como uma forma de proporcionar uma melhor simetria para a semântica de provas e uma maneira de estudar a complexidade das provas proposicionais e fornecer provadores de teoremas mais eficientes, em relação ao tamanho das provas proposicionais. O objetivo deste trabalho é reduzir o peso/tamanho de deduções. Apresentamos formalismos de grafos de prova que visam capturar a estrutura lógica de uma dedução e uma forma de facilitar a visualização das propriedades. A vantagem destes formalismos é que as fórmulas e sub-deduções em dedução natural, preservadas na estrutura de grafo, podem ser compartilhadas eliminando sub-deduções desnecessárias resultando na prova reduzida. Neste trabalho, damos uma definição precisa de grafos de prova para a lógica puramente implicacional, logo estendemos esse resultado para a lógica proposicional completa e mostramos como reduzir (eliminando fórmulas máximas) essas representações de tal forma que um teorema de normalização pode ser provado através da contagem do número de fórmulas máximas na derivação original. A normalização forte será uma consequência direta desta normalização, uma vez que qualquer redução diminui as medidas correspondentes da complexidade da derivação. Continuando com o nosso objetivo de estudar a complexidade das provas, a abordagem atual também fornece representações de grafo para lógica de primeira ordem, a inferência profunda e lógica bi-intuicionista.

## Palavras-chave

Teoria da Prova; Grafos de Prova; Dedução Natural; Complexidade de Provas; Normalização Forte; Lógica Clássica; Lógica Minimal Implicacional; Inferência Profunda; Lógica Bi-intuicionista;

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