Referências Bibliográficas

[1] ALVIN, M. **Esqueletos Afins e Equações de Propagação.** Tese de Doutorado, IMPA, 2005. Orientado por Ralph Teixeira e Luiz Velho.

[2] ANDRADE, M.; CABRAL, A.; MELLO, V.; PEIXOTO, A. ; LEWINER, T. **Curvas e Superfícies Implícitas: Noções de Geometrias Diferencial e Discreta.** I Colóquio de Matemática da Região Nordeste, 2011.

[3] ANDRADE, M.; LEWINER, T. **Affine- invariant curvature estimators for implicit surfaces**. Computer Aided Geometric Design (submetido), 2011.

[4] ANDRADE, M.; LEWINER, T. **Affine-invariant estimators for implicit surfaces**. Sibgrapi (submetido), 2011.

[5] ANDRADE, M.; LEWINER, T. **Cálculo e Estimação de Invariantes Geométricos: Uma Introdução às Geometrias Euclidiana e Afim**. 28º Colóquio Brasileiro de Matemática, 2011.

[6] ARBERT, K.; SNYDER, W.; BURKHARDT, H. ; HIRZINGER, G. **Application of affine-invariant Fourier descriptors to recognition of 3-D objects**. Transactions on Pattern Analysis an Machine Intelligence, 12(7):640–647, 1990.

[7] BETELU, S.; SAPIRO, G. ; TANNENBAUM, A. **Affine invariant erosion of 3d shapes**. In: INTERNATIONAL CONFERENCE ON COMPUTER VISION, volume 2, p. 174–180. IEEE, 2002.

[8] BUCHIN, S. **Affine differential geometry**. Routledge, 1983.

[9] CABRAL, A.; MELLO, V. **Visualização da curvatura de objetos implícitos em um sistema extensível.** Dissertação de Mestrado, UFAL, 2010. Orientado por Vinícius Mello.

[10] CALABI, E. **Hypersurfaces with maximal affinely invariant area**. American Journal of Mathematics, 104(1):91–126, 1982.

[11] COHEN-STEINER, D.; MORVAN, J.-M. **Restricted delaunay triangulations and normal cycle**. Proceedings of the 19th Annual Symposium on Computational Geometry, p. 312–321, 2003.

[12] CRAIZER, M.; ALVIM, M. ; TEIXEIRA, R. **Area distances of convex plane curves and improper affine spheres**. SIAM Journal on Mathematical Imaging, (1(3)):209–227, 2008.

[13] CRAIZER, M.; LEWINER, T. ; MORVAN, J. **Combining points and tangents into parabolic polygons**. Journal Mathematical Imaging Vision, 29:131–140, 2007.

[14] DAVIS, D. **Affine Differential Geometry & Singularity Theory.** Tese de Doutorado, University of Liverpool, 2008. Orientado por Peter Giblin.

[15] DO CARMO, M. **Differential geometry of curves and surfaces**. Prentice Hall, 1976.

[16] FARIN, G.; HOSCHEK, J. ; KIM, M. **Handbook of computer aided geometric design**. North-Holland, 2002.

[17] FORSYTH, D.; PONCE, J. **Computer vision: a modern approach**. Prentice Hall, 2002.

[18] FOULONNEAU, A.; CHARBONNIER, P. ; HEITZ, F. **Affine-invariant geometric shape priors for region-based active contours**. Transactions on Pattern Analysis an Machine Intelligence, p. 1352–1357, 2006.

[19] GAL, R.; COHEN-OR, D. **Salient geometric features for partial shape matching and similarity**. Transactions on Graphics, 25(1):130–150, 2006.

[20] GAL, R.; SHAMIR, A.; HASSNER, T.; PAULY, M. ; COHEN-OR, D. **Surface reconstruction using local shape priors**. In: SYMPOSIUM ON GEOMETRY PROCESSING, p. 253–262. Eurographics, 2007.

[21] GOLDMAN, R. **Curvature formulas for implicit curves and surfaces**. Computer Aided Geometric Design, 22(7):632–658, 2005.

[22] GOMES, J.; VELHO, L. **Fundamentos de Computação Gráfica**. IMPA, 2003.

[23] LEBEAU, G. **Théorie das distributions et analyse de Fourier**. Ecole Polytechnique, Département de Mathématiques, 1998.

[24] LEWINER, T.; GOMES, J.; LOPES, H. ; CRAIZER, M. **Curvature and torsion estimators based on parametric curve fitting**. Computers & Graphics, 29(5):641–655, 2005.

[25] LORENSEN, W.; CLINE, H. **Marching Cubes: A high resolution 3D surface construction algorithm**. In: SIGGRAPH, p. 169. ACM, 1987.

[26] LOWE, D. **Distinctive image features form scale invariant keypoints**. International Journal of computer Vision, 60(2):91–110, 2004.

[27] MIKOLAJCZYK, K.; SCHMID, C. **A perfomance evaluation of local descriptors**. Transactions on Pattern Analysis and Machine Intelligence, 27(7):1615–1630, 2005.

[28] MIKOLAJCZYK, K.; SCHMID, C. **A comparison of affine region detectors**. International Journal of Computer Vision, 65(1):43–72, 2006.

[29] NOMIZU, K.; SASAKI, T. **Affine differential geometry: geometry of affine immersions**. Cambridge University Press, 1994.

[30] NÜRNBERGER, G.; SCHMIDT, J. ; WALZ, G. **Multivariate approximation and splines**. Birkhauser, 1997.

[31] OHBUCHI, R.; OSADA, K.; FURUKA, T. ; BANNO, T. **Salient local visual features for shape-based 3d model retrieval**. Proc. IEEE International Conference on Shape Modeling and Applications, 2008.

[32] PAIVA, A.; PETRONETTO, F.; LEWINER, T. ; TAVARES, G. **Particle-based non-newtonian fluid animation for melting objects**. Sibgrapi, p. 78–85, 2006.

[33] RAVIV, D.; BRONSTEIN, A.; BRONSTEIN, M.; KIMMEL, R. ; SOCHEN, N. **Affine-invariant geodesic geometry of deformable 3d shapes**. Computers & Graphics, 35(3):692–697, 2011.

[34] REUTER, M.; WOLTER, F. ; PEINECKE, N. **Laplace-Beltrami spectra as shape-DNA of surfaces and solids**. Computer-Aided Design, 38(4):342–366, 2006.

[35] ROTHGANGER, F.; LAZEBNIK, S.; SCHMID, C. ; PONCE, J. **3d object modeling and recognition using local affine-invariant image descriptors and multi-view spatial constraints**. International Journal of Computer Vision, 66(3):231–259, 2006.

[36] SAPIRO, G.; TANNENBAUM, A. **Affine invariant scale-space**. International Journal of Computer Vision, 11(1):25–44, 1993.

[37] SPIVAK, M. **A Compreensive Introduction to Differential Geometry**. Publish or Perish, 1999.

[38] VELHO, L.; GOMES, J. ; FIGUEIREDO, L. H. **Implicit Objects in Computer Graphics**. Springer, 2002.

[39] ZULIANI, M.; BERTELLI, L.; KENNEY, C.; CHANDRASEKARAN, S. ; MANJUNATH, B. **Drums, curve descriptors and affine invariant region matching**. International Journal of computer Vision, 26:347–360, 2008.

A
Cálculo dos Invariantes Diretamente

Este apêndice tem por objetivo escrever os invariantes geométricos de uma superfície implícita $S = \{(x, y, z) \in \mathbb{R}^3 / f(x, y, z) = 0\}$, onde f é de classe C^4 e 0 é valor regular de f, a partir da própria função f e reforçar a ideia que isto inclui um longo cálculo. Sabemos que toda superfície regular pode ser vista localmente como um gráfico $\mathcal{G} = \{(x, y, g(x, y)/(x, y) \in U\}$, em particular na seção 3.1 encontramos o plano tangente, a métrica afim escrita em termos da função f. Outros elementos geométricos importantes são o vetor normal $\mathbf{N}_e$ e a curvatura K_e que são dados, respectivamente, por

$$
\begin{aligned}
\mathbf{N}_e &= \frac{(-g_x, -g_y, 1)}{\sqrt{g_x^2 + g_y^2 + 1}} = (f_z^2 + f_x^2 + f_y^2)^{-1/2}(f_x, f_y, f_z) \\
K_e &= \frac{\left(f_{zz}f_{yy} - f_{yz}^2\right) f_x^2 + (-2f_{xy}f_{zz} + 2f_{xz}f_{yz}) f_y f_x}{\left(f_z^2 + f_x^2 + f_y^2\right)^2} \\
&+ \frac{2\left(-f_{xz}f_{yy} + f_{xy}f_{yz}\right) f_x f_z + \left(f_{xx}f_{zz} - f_{xz}^2\right) f_y^2}{\left(f_z^2 + f_x^2 + f_y^2\right)^2} \\
&+ \frac{-2\left(-f_{xz}f_{xy} + f_{xx}f_{yz}\right) f_y f_z + \left(f_{xx}f_{yy} - f_{xy}^2\right) f_z^2}{\left(f_z^2 + f_x^2 + f_y^2\right)^2}.
\end{aligned}
$$

Notamos que é preciso fazer um escalonamento do vetor normal $\mathbf{N}_e$ usando a curvatura K_e para obtermos um vetor contravariante o co-normal ν (ver seção 3.2) cuja fórmula em função de f é

$$
\nu = \frac{1}{f_z \mathsf{d}^{1/4}} \left(f_x, f_y, f_z \right),
$$

onde

$$
\begin{aligned}
\mathsf{d} = \frac{1}{f_z^4} \cdot \Big(& \left(f_{yy}f_{zz} - f_{yz}^2\right) f_x^2 + 2\left(f_{xz}f_{xy} - f_{xx}f_{yz}\right) f_y f_z + \\
& \left(f_{zz}f_{xx} - f_{xz}^2\right) f_y^2 + 2\left(f_{xy}f_{yz} - f_{yy}f_{xz}\right) f_z f_x + \\
& \left(f_{xx}f_{yy} - f_{xy}^2\right) f_z^2 + 2\left(f_{yz}f_{xz} - f_{zz}f_{xy}\right) f_x f_y \Big).
\end{aligned} \tag{A-1}
$$

Encontraremos a expressão do normal afim a partir da função f.

Utilizando os cálculos e definições da seção 3.2, temos as componentes do vetor normal afim em função das derivadas da função g. Usando a regra da cadeia obtemos as expressões das derivadas da função g até a terceira ordem. Substituindo estas expressões nas fórmulas explícitas de ξ dadas na seção 3.2, obtemos as coordenadas de ξ

$$
\begin{aligned}
\xi_1 = a\Big[\, & f_x^2\ (-f_z f_{xy} f_{zz} f_{yyy} - 2 f_y f_{yz} f_{yyz} f_{xz} + f_y f_{zz} f_{xz} f_{yyy} + f_{yy} f_y f_{yzz} f_{xz} \\
- \ & f_z f_{yy} f_{xy} f_{yzz} - f_{yy} f_y f_{xzz} f_{yz} - 4 f_{zz} f_{xz} f_{yy}^2 - 4 f_{xy} f_{yz}^3 + 4 f_{yy} f_{xz} f_{yz}^2 \\
+ \ & 2 f_z f_{xy} f_{yz} f_{yyz} + 4 f_{yy} f_{xy} f_{yz} f_{zz} - 2 f_z f_{yy} f_{yz} f_{xyz} - f_y f_{zz} f_{xyy} f_{yz} \\
+ \ & 2 f_y f_{yz}^2 f_{xyz} + f_z f_{yy}^2 f_{xzz} + f_z f_{yy} f_{zz} f_{xyy}\big) \\
+ \ & f_x\ (8 f_{yy} f_{xz} f_{xy} f_y f_{zz} - 2 f_{xx} f_{yy} f_y f_{zz} f_{yz} - 2 f_z f_{yy} f_y f_{zz} f_{xxy} + 4 f_z f_{yy} f_y f_{xxz} f_{yz} \\
+ \ & 2 f_z f_{xyy} f_{xy} f_y f_{zz} + 4 f_z f_y f_{xz} f_{xyy} f_{yz} - 4 f_z f_y f_{xy} f_{xyz} f_{yz} - 2 f_z f_{yy} f_{xy} f_y f_{xzz} \\
- \ & 12 f_z f_{yy} f_{xy} f_{xz} f_{yz} - 2 f_y^2 f_{zz} f_{xyy} f_{xz} - 2 f_z^2 f_{yy} f_{xz} f_{xyy} - 2 f_z^2 f_{xyy} f_{xy} f_{yz} \\
+ \ & 4 f_z^2 f_{yy} f_{xy} f_{xyz} + 2 f_{xy} f_y^2 f_{xzz} f_{yz} + 2 f_z^2 f_{xy} f_{xz} f_{yyy} - 2 f_z f_y f_{xz}^2 f_{yyy} \\
- \ & 2 f_{xy} f_{yzz} f_y^2 f_{xz} - 6 f_{yy} f_{xz}^2 f_{yz} f_y + 2 f_z f_{xx} f_{zz} f_{yy}^2 - 2 f_z f_{yy} f_{xy}^2 f_{zz} \\
- \ & 6 f_y f_{zz} f_{yz} f_{xy}^2 - 2 f_z f_y f_{xxy} f_{yz}^2 + 2 f_y^2 f_{xxy} f_{yz} f_{zz} + 2 f_z^2 f_{yy} f_{xxy} f_{yz} \\
+ \ & 4 f_y f_{xz} f_{yz}^2 f_{xy} - 2 f_z f_{xx} f_{yy} f_{yz}^2 - 2 f_z^2 f_{xxz} f_{yy}^2 + 2 f_{xx} f_y f_{yz}^3 \\
+ \ & 2 f_y^2 f_{xz}^2 f_{yyz} - 2 f_y^2 f_{xxz} f_{yz}^2 + 6 f_z f_{yy}^2 f_{xz}^2 - 2 f_z^2 f_{xy}^2 f_{yyz} + 8 f_z f_{xy}^2 f_{yz}^2\big) \\
+ \ & f_y^3\ (2 f_{xz} f_{xxz} f_{yz} - 2 f_{xyz} f_{xz}^2 + f_{xxy} f_{xz} f_{zz} - f_{xx} f_{xzz} f_{yz} - f_{xxx} f_{zz} f_{yz} + f_{xx} f_{yzz} f_{xz}) \\
+ \ & f_y^2\ (f_z f_{xx} f_{yy} f_{xzz} - 4 f_z f_{xxy} f_{xz} f_{yz} - 2 f_z f_{xx} f_{yyz} f_{xz} - 2 f_z f_{xy} f_{xxz} f_{yz} \\
- \ & f_z f_{xx} f_{xy} f_{yzz} - 2 f_{xx} f_{yy} f_{xz} f_{zz} - 2 f_z f_{yy} f_{xz} f_{xxz} + 4 f_{xx} f_{xy} f_{yz} f_{zz} \\
+ \ & f_z f_{yy} f_{xxx} f_{zz} - 2 f_{xx} f_{yz}^2 f_{xz} + 4 f_z f_{xyz} f_{xy} f_{xz} - 2 f_{xy}^2 f_{xz} f_{zz} \\
+ \ & 2 f_z f_{xxx} f_{yz}^2 + 2 f_z f_{xyy} f_{xz}^2 + 2 f_{yy} f_{xz}^3 - f_z f_{xy} f_{xxy} f_{zz} + 2 f_z f_{xx} f_{xyz} f_{yz}\big) \\
+ \ & f_y\ \big(-f_z^2 f_{xx} f_{xyy} f_{yz} + 8 f_z f_{xx} f_{yy} f_{xz} f_{yz} + 4 f_z f_{xy}^2 f_{xz} f_{yz} - 6 f_z f_{yy} f_{xy} f_{xz}^2 \\
+ \ & f_z^2 f_{xx} f_{yyy} f_{xz} + 2 f_z^2 f_{xx} f_{xy} f_{yyz} - 2 f_z f_{xx} f_{yy} f_{xy} f_{zz} + 4 f_z^2 f_{xy} f_{yz} f_{xxy} \\
- \ & 3 f_z^2 f_{yy} f_{xxx} f_{yz} - 2 f_z^2 f_{xy}^2 f_{xyz} + 2 f_z f_{xy}^3 f_{zz} + 3 f_z^2 f_{yy} f_{xxy} f_{xz} \\
- \ & 2 f_z^2 f_{xx} f_{yy} f_{xyz} + 2 f_z^2 f_{yy} f_{xy} f_{xxz} - 4 f_z^2 f_{xy} f_{xyy} f_{xz} - 6 f_z f_{xx} f_{xy} f_{yz}^2\big) \\
+ \ & 4 f_z^2 f_{xx} f_{yy} f_{xy} f_{yz} + 2 f_z^3 f_{xy}^2 f_{xyy} + 4 f_z^2 f_{yy} f_{xz} f_{xy}^2 - f_z^3 f_{xx} f_{xy} f_{yyy} \\
- \ & 3 f_z^3 f_{xy} f_{xxy} f_{yy} - 4 f_z^2 f_{xy}^3 f_{yz} + f_z^3 f_{xxx} f_{yy}^2 + f_z^3 f_{xx} f_{xyy} f_{yy} - 4 f_z^2 f_{xx} f_{xz} f_{yy}^2 \Big],
\end{aligned}
$$

onde

$$a = (4(f_z^3 \mathsf{d}^{3/4})\,(-f_{xz} f_{yy} + f_{xy} f_{yz})\, f_x + f_z\,\big(f_{xx} f_{yy} - f_{xy}^2\big) + (-f_{xx} f_{yz} + f_{xy} f_{xz})\, f_y)^{-1},$$

$$
\begin{aligned}
\xi_2 = a\Big[\, & f_x^3\ \big(-2f_{xyz}f_{yz}^2+f_{zz}f_{xyy}f_{yz}+f_{yy}f_{xzz}f_{yz}+2f_{xz}f_{yz}f_{yyz}-f_{xz}f_{zz}f_{yyy}-f_{yy}f_{xz}f_{yzz}\big)\\
+\ & f_x^2\ \big(-2f_{xx}f_{yz}f_{zz}f_{yy}+4f_zf_{xyz}f_{xy}f_{yz}-2f_{xxy}f_{yz}f_yf_{zz}+4f_{yy}f_{xz}f_{xy}f_{zz}\\
& -\ 2f_{xzz}f_{xy}f_{yz}f_y+2f_{xz}f_yf_{zz}f_{xyy}+f_zf_{xx}f_{yy}f_{yzz}-2f_zf_{yy}f_{xxz}f_{yz}\\
& -\ 2f_{yy}f_{xz}^2f_{yz}-2f_zf_{xx}f_{yz}f_{yyz}-f_zf_{yy}f_{xzz}f_{xy}+2f_zf_{yy}f_{xz}f_{xyz}+2f_{xx}f_{yz}^3\\
& -\ 2f_zf_{xz}f_{xy}f_{yyz}+f_zf_{xx}f_{zz}f_{yyy}-2f_{xy}^2f_{yz}f_{zz}+2f_{xxz}f_yf_{yz}^2-2f_{xz}^2f_{yyz}f_y\\
& +\ 2f_zf_{xz}^2f_{yyy}-4f_zf_{xz}f_{xyy}f_{yz}+2f_{xz}f_{xy}f_yf_{yzz}+2f_zf_{yz}^2f_{xxy}-f_zf_{zz}f_{xyy}f_{xy}\big)\\
+\ & f_x\ \ \big(2f_{yy}f_{xz}^3f_y+2f_zf_{xy}^3f_{zz}-2f_z^2f_{xy}^2f_{xyz}+2f_{xz}^2f_{xyz}f_y^2-2f_zf_{xz}^2f_yf_{xyy}\\
& +\ 4f_z^2f_{xz}f_{xy}f_{xyy}+3f_z^2f_{xx}f_{xyy}f_{yz}-2f_z^2f_{xx}f_{yy}f_{xyz}+2f_zf_{xy}^2f_yf_{xzz}\\
& +\ f_{xx}f_{xzz}f_y^2f_{yz}-3f_z^2f_{xx}f_{xz}f_{yyy}+2f_z^2f_{xx}f_{xy}f_{yyz}-f_{xx}f_{xz}f_{yzz}f_y^2\\
& -\ 6f_{xz}f_yf_{xy}^2f_{zz}+f_{xxx}f_{yz}f_y^2f_{zz}-2f_zf_{xxx}f_yf_{yz}^2+f_z^2f_{yy}f_{xxx}f_{yz}\\
& -\ f_{xz}f_{xxy}f_y^2f_{zz}-f_z^2f_{yy}f_{xz}f_{xxy}-4f_z^2f_{xxy}f_{yz}f_{xy}+2f_z^2f_{yy}f_{xxz}f_{xy}\\
& -\ 2f_y^2f_{xz}f_{xxz}f_{yz}-6f_zf_{yy}f_{xz}^2f_{xy}-6f_zf_{xx}f_{yz}^2f_{xy}-6f_{xx}f_{xz}f_{yz}^2f_y\\
& +\ 4f_zf_{xz}f_{xy}^2f_{yz}+4f_{xz}^2f_yf_{yz}f_{xy}-2f_zf_{xx}f_{yy}f_{zz}f_{xy}-2f_{xx}f_{yy}f_{xz}f_yf_{zz}\\
& +\ 8f_{xx}f_{yz}f_{xy}f_yf_{zz}+2f_zf_{xxy}f_{xy}f_yf_{zz}+4f_zf_{xz}f_yf_{xxy}f_{yz}-2f_zf_{xx}f_{xyy}f_yf_{zz}\\
& -\ 4f_zf_{xz}f_yf_{xy}f_{xyz}+4f_zf_{xx}f_{xz}f_{yyz}f_y-2f_zf_{xx}f_{xy}f_yf_{yzz}+8f_zf_{xx}f_{yy}f_{yz}f_{xz}\big)\\
+\ & f_y^2\ \big(-f_zf_{xx}f_{xzz}f_{xy}+4f_{xx}f_{yz}f_{xz}^2+4f_{xx}f_{xz}f_{xy}f_{zz}-2f_zf_{xx}f_{xz}f_{xyz}\\
& -\ f_zf_{xxx}f_{zz}f_{xy}+2f_zf_{xz}f_{xxz}f_{xy}+f_zf_{xx}f_{xxy}f_{zz}-4f_{yz}f_{zz}f_{xx}^2\\
& +\ f_zf_{xx}^2f_{yzz}-4f_{xz}^3f_{xy}\big)\\
+\ & f_y\ \ \big(-2f_z^2f_{xx}^2f_{yyz}+2f_z^2f_{xx}f_{xyy}f_{xz}+2f_zf_{xx}^2f_{zz}f_{yy}-2f_zf_{xx}f_{xy}^2f_{zz}\\
& -\ 2f_z^2f_{xxy}f_{xy}f_{xz}+2f_z^2f_{xxx}f_{yz}f_{xy}-12f_zf_{xx}f_{xz}f_{xy}f_{yz}+4f_z^2f_{xx}f_{xyz}f_{xy}\\
+\ & 6f_zf_{xx}^2f_{yz}^2-2f_z^2f_{xx}f_{yz}f_{xxy}-2f_zf_{xx}f_{yy}f_{xz}^2-2f_z^2f_{xy}^2f_{xxz}+8f_zf_{xz}^2f_{xy}^2\big)\\
-\ & 4f_z^2f_{xz}f_{xy}^3-3f_z^3f_{xx}f_{xyy}f_{xy}+2f_z^3f_{xy}^2f_{xxy}+4f_z^2f_{xx}f_{yy}f_{xy}f_{xz}\\
+\ & 4f_z^2f_{xx}f_{yz}f_{xy}^2-f_z^3f_{xxx}f_{xy}f_{yy}+f_z^3f_{yyy}f_{xx}^2+f_z^3f_{xx}f_{xxy}f_{yy}-4f_z^2f_{xx}^2f_{yz}f_{yy}.
\end{aligned}
$$

E finalmente,

$$
\begin{aligned}
\xi_3 = a\Big[\, & f_x^3\ \big(-f_{yy}^2f_{xzz}+2f_{yy}f_{yz}f_{xyz}-f_{yy}f_{zz}f_{xyy}-2f_{xy}f_{yz}f_{yyz}+f_{xy}f_{zz}f_{yyy}+f_{yy}f_{xy}f_{yzz}\big)\\
+\ & f_x^2\ \big(2f_zf_{yy}f_{xz}f_{xyy}+2f_{yy}^2f_{xz}^2+2f_{xx}f_yf_{yz}f_{yyz}+2f_zf_{xyy}f_{xy}f_{yz}\\
& -\ 4f_zf_{yy}f_{xy}f_{xyz}+2f_{xz}f_yf_{xy}f_{yyz}-2f_{yy}f_{xz}f_{xyz}f_y-f_{zz}f_yf_{xy}f_{xyy}\\
& -\ f_{xx}f_yf_{yzz}f_{yy}+4f_{xy}^2f_{yz}^2-f_{xx}f_yf_{zz}f_{yyy}-2f_{yy}f_{xxz}f_{yz}f_y-2f_zf_{xy}f_{xz}f_{yyy}\\
& +\ 3f_{yy}f_{xy}f_yf_{xzz}-2f_zf_{yy}f_{xxy}f_{yz}+2f_zf_{xxz}f_{yy}^2-2f_{xx}f_{yy}f_{yz}^2-2f_{yy}f_{xy}^2f_{zz}\\
& +\ 2f_{yy}f_{xxy}f_yf_{zz}-2f_{xy}^2f_yf_{yzz}+2f_{xx}f_{zz}f_{yy}^2+2f_zf_{xy}^2f_{yyz}-4f_{yy}f_{xz}f_{yz}f_{xy}\big)\\
+\ & f_x\ (2f_zf_{yy}f_{xxx}f_{yz}f_y-2f_zf_{yy}f_yf_{xxy}f_{xz}-4f_zf_{yy}f_{xy}f_yf_{xxz}-2f_zf_{xx}f_{xyy}f_{yz}f_y\\
& +\ 4f_zf_{xx}f_{yy}f_{xyz}f_y+2f_zf_{xx}f_{xz}f_yf_{yyy}-4f_zf_{xx}f_yf_{xy}f_{yyz}+8f_zf_{xx}f_{yy}f_{xy}f_{yz}
\end{aligned}
$$

$$
\begin{aligned}
&& + \quad & 8f_{xx}f_{yy}f_{xz}f_{yz}f_y - 4f_{xx}f_{yy}f_{xy}f_yf_{zz} - 8f_zf_{xy}^3f_{yz} + 4f_{xy}^3f_yf_{zz} \\
&& - \quad & 2f_z^2f_{xyy}f_{xy}^2 - 2f_{xzz}f_y^2f_{xy}^2 + 4f_zf_{xy}^2f_{xyz}f_y - 2f_{xx}f_{xyz}f_{yz}f_y^2 \\
&& + \quad & f_z^2f_{xx}f_{xy}f_{yyy} - f_{xx}f_{yy}f_{xzz}f_y^2 - 2f_{xx}f_{xz}f_{yyz}f_y^2 + 3f_{xx}f_{xy}f_y^2f_{yzz} \\
&& - \quad & f_{yy}f_{xxx}f_{zz}f_y^2 - f_{xy}f_y^2f_{zz}f_{xxy} + 3f_z^2f_{yy}f_{xy}f_{xxy} + 2f_{yy}f_y^2f_{xz}f_{xxz} \\
&& + \quad & 2f_{xy}f_y^2f_{xxz}f_{yz} + 2f_{xx}f_{xyy}f_y^2f_{zz} - f_z^2f_{xx}f_{yy}f_{xyy} - 8f_zf_{xx}f_{xz}f_{yy}^2 \\
&& - \quad & 4f_{yy}f_{xy}f_yf_{xz}^2 + 8f_zf_{yy}f_{xz}f_{xy}^2 - 4f_{xx}f_{xy}f_{yz}^2f_y - f_z^2f_{xxx}f_{yy}^2\big) \\
& + \quad f_y^3 & (&-f_{xx}f_{xxy}f_{zz}+f_{xxx}f_{zz}f_{xy}+2f_{xx}f_{xz}f_{xyz}+f_{xx}f_{xzz}f_{xy}-f_{xx}^2f_{yzz} \\
& - \quad & & 2f_{xz}f_{xxz}f_{xy}) \\
& + \quad f_y^2 & \big(&4f_{xy}^2f_{xz}^2 - 2f_zf_{xx}f_{xyy}f_{xz} - 4f_zf_{xx}f_{xy}f_{xyz} + 2f_zf_{xx}f_{xxy}f_{yz} \\
&& + \quad & 2f_zf_{xy}^2f_{xxz}2f_{xx}^2f_{yz}^2 + 2f_zf_{xxy}f_{xy}f_{xz} - 4f_{xx}f_{xz}f_{xy}f_{yz} \\
&& + \quad & 2f_zf_{xx}^2f_{yyz} - 2f_{xx}f_{xz}^2f_{yy} + 2f_{xx}^2f_{zz}f_{yy} - 2f_{xx}f_{xy}^2f_{zz} - 2f_zf_{xxx}f_{yz}f_{xy}\big) \\
& + \quad f_y & \big(&-f_z^2f_{xx}f_{yy}f_{xxy} + f_z^2f_{yy}f_{xxx}f_{xy} + 3f_z^2f_{xx}f_{xy}f_{xyy} + 8f_zf_{xx}f_{yy}f_{xy}f_{xz} \\
&& - \quad & 8f_zf_{xy}^3f_{xz} - 8f_zf_{xx}^2f_{yz}f_{yy} - f_z^2f_{xx}^2f_{yyy} + 8f_zf_{xx}f_{xy}^2f_{yz} - 2f_z^2f_{xy}^2f_{xxy}\big) \\
& + \quad 4 & & f_z^2f_{xx}^2f_{yy}^2 + 4f_z^2f_{xy}^4 - 8f_z^2f_{xx}f_{yy}f_{xy}^2.
\end{aligned}
$$

Notemos que ao substituirmos $f_x = 0 = f_y = f_{xy}$ e $f_z = 1$ em ξ temos exatamente a expressão do ξ dada no capítulo 4.

Agora, queremos calcular as expressões para as curvaturas afins, para isso é necessário calcularmos as derivadas do co-normal afim ν e do normal afim ξ. Denotaremos por $\nu_x = (\partial_x\nu_1, \partial_x\nu_2, \partial_x\nu_3)$, $\nu_y = (\partial_y\nu_1, \partial_y\nu_2, \partial_y\nu_3)$ e $\xi_x = (\partial_x\xi_1, \partial_x\xi_2, \partial_x\xi_3)$, $\xi_y = (\partial_y\xi_1, \partial_y\xi_2, \partial_y\xi_3)$. Obtemos,

$$
\begin{aligned}
\partial_x\nu_1 &= \frac{1}{4\mathsf{d}^{5/4}} \cdot (g_xg_{xxx}g_{yy} + g_xg_{xx}g_{xyy} - 2g_xg_{xy}g_{xxy} - 4g_{xx}^2g_{yy} + 4g_{xx}g_{xy}^2), \\
\partial_x\nu_2 &= \frac{1}{4\mathsf{d}^{5/4}} \cdot (g_yg_{xxx}g_{yy} + g_yg_{xx}g_{xyy} - 2g_yg_{xy}g_{xxy} - 4g_{xy}g_{xx}g_{yy} + 4g_{xy}^3), \\
\partial_x\nu_3 &= \frac{1}{4\mathsf{d}^{5/4}} \cdot (-g_{xxx}g_{yy} - g_{xx}g_{xyy} + 2g_{xy}g_{xxy})
\end{aligned}
$$

e

$$
\begin{aligned}
\partial_y\nu_1 &= \frac{1}{4\mathsf{d}^{5/4}} \cdot (g_xg_{xxy}g_{yy} + g_xg_{xx}g_{yyy} - 2g_xg_{xy}g_{xyy} - 4g_{xy}g_{xx}g_{yy} + 4g_{xy}^3), \\
\partial_y\nu_2 &= \frac{1}{4\mathsf{d}^{5/4}} \cdot (g_yg_{xxy}g_{yy} + g_yg_{xx}g_{yyy} - 2g_yg_{xy}g_{xyy} - 4g_{xx}g_{yy}^2 + 4g_{yy}g_{xy}^2), \\
\partial_y\nu_3 &= -\frac{1}{4\mathsf{d}^{5/4}} \cdot (g_{xxy}g_{yy} + g_{xx}g_{yyy} - 2g_{xy}g_{xyy}),
\end{aligned}
$$

onde as expressões de g_x, g_y, g_{xx}, g_{xy} e g_{yy} foram dadas no capítulo 4 e as demais derivadas são

$$
\begin{aligned}
g_{xxx} &= -\frac{f_{xxx}}{f_z} + \frac{3(f_{xxz}f_x + f_{xz}f_{xx})}{f_z^2} + \frac{3(-f_{zz}f_xf_{xx} - 2f_{xz}^2f_x - f_{xzz}f_x^2)}{f_z^3} \\
&+ \frac{9f_{xz}f_{zz}f_x^2f_x^3f_{zzz}}{f_z^4} - \frac{3f_{zz}^2f_x^3}{f_z^5} \\
g_{xxy} &= -\frac{f_{xxy}}{f_z} + \frac{f_{yz}f_{xx} + 2f_xf_{xyz} + 2f_{xz}f_{xy} + f_{xxz}f_y}{f_z^2} \\
&+ \frac{-2f_{xz}^2f_y - f_x^2f_{yzz} - f_{zz}f_yf_{xx} - 2f_{zz}f_xf_{xy} - 4f_{xz}f_{yz}f_x - 2f_xf_{xzz}f_y}{f_z^3} \\
&+ \frac{f_x^2f_{zzz}f_y + 6f_{xz}f_yf_{zz}f_x + 3f_{zz}f_x^2f_{yz}}{f_z^4} - \frac{3f_{zz}^2f_x^2f_y}{f_z^5} \\
g_{xyy} &= -\frac{f_{xyy}}{f_z} + \frac{2f_{yz}f_{xy} + f_xf_{yyz} + f_{xz}f_{yy} + 2f_{xyz}f_y}{f_z^2} \\
&- \frac{4f_{yz}f_yf_{xz} + 2f_{zz}f_yf_{xy} + f_{xzz}f_y^2 + 2f_xf_{yzz}f_y + 2f_{yz}^2f_x + f_{zz}f_xf_{yy}}{f_z^3} \\
&+ \frac{3f_{xz}f_{zz}f_y^2 + 6f_{yz}f_yf_{zz}f_x + f_y^2f_xf_{zzz}}{f_z^4} - \frac{3f_{zz}^2f_xf_y^2}{f_z^5} \\
g_{yyy} &= -\frac{f_{yyy}}{f_z} + \frac{3f_{yyz}f_y + 3f_{yz}f_{yy}}{f_z^2} + \frac{-3f_{zz}f_yf_{yy} - 6f_{yz}^2f_y - 3f_{yzz}f_y^2}{f_z^3} \\
&+ \frac{9f_{yz}f_{zz}f_y^2 + f_y^3f_{zzz}}{f_z^4} - \frac{3f_{zz}^2f_y^3}{f_z^5}.
\end{aligned}
$$

As derivadas do normal afim com relação a x e y são

$$
\begin{aligned}
\partial_x\xi_1 = \frac{1}{16\mathsf{d}^{5/4}} \big(&-12g_{xy}^3g_{xxxy}g_{yy} + 3g_{xx}^2g_{xyy}^2g_{yy} - 39g_{xx}g_{xyy}g_{yy}g_{xy}g_{xxy} \\
&- 3g_{xx}g_{xy}g_{yyy}g_{xxx}g_{yy} + 7g_{xxx}^2g_{yy}^3 + 8g_{xy}^4g_{xxyy} + 2g_{xx}g_{xyy}g_{yy}^2g_{xxx} \\
&- 7g_{xx}^2g_{xy}g_{yyy}g_{xyy} + 10g_{xx}^2g_{xy}g_{yyy}g_{xxy} - 35g_{xxx}g_{yy}^2g_{xy}g_{xxy} \\
&+ 26g_{xy}^2g_{xyy}g_{xxx}g_{yy} - 4g_{xx}g_{xxyy}g_{yy}g_{xy}^2 + 4g_{xx}^2g_{xxy}g_{yyy}g_{yy} \\
&+ 4^2g_{xx}g_{xy}g_{xyyy}g_{yy} + 12g_{xy}g_{xxxy}g_{yy}^2g_{xx} + 30g_{xy}^2g_{xxy}^2g_{yy} \\
&+ 18g_{xy}^2g_{xyy}^2g_{xx} - 24g_{xy}^3g_{xyy}g_{xxy} - 4g_{xx}^2g_{xxyy}g_{yy}^2 - 4g_{xxx}g_{xy}^3g_{yyy} \\
&- 4g_{xx}g_{xy}^3g_{xyyy} - 4g_{xxxx}g_{yy}^3g_{xx} + 4g_{xxxx}g_{yy}^2g_{xy}^2 + 12g_{xxy}^2g_{yy}^2g_{xx}\big)\,, \\
\partial_x\xi_2 = \frac{1}{16\mathsf{d}^{5/4}} \big(&-7g_{xy}g_{xxx}^2g_{yy}^2 - 12g_{xy}^3g_{xx}g_{xxyy} - 12g_{xxy}^2g_{xy}^3 + 8g_{xxxy}g_{xy}^4 \\
&- 21g_{xy}g_{xx}^2g_{xyy}^2 + 7g_{xx}^3g_{yyy}g_{xyy} - 4g_{xy}^3g_{xxxx}g_{yy} - 16g_{xy}^3g_{xxx}g_{xyy} \\
&- 4g_{xx}^3g_{xyyy}g_{yy} + 4g_{xx}^2g_{xyyy}g_{xy}^2 - 4g_{xxxy}g_{xx}^2g_{yy}^2 + 28g_{xxy}g_{xy}^2g_{xxx}g_{yy} \\
&+ 48g_{xxy}g_{xy}^2g_{xx}g_{xyy} - g_{xx}^2g_{yyy}g_{xxx}g_{yy} - 14g_{xx}^2g_{yyy}g_{xy}g_{xxy} + 7g_{xxy}g_{xx}g_{yy}^2g_{xxx} \\
&+ 15g_{xxy}g_{xx}^2g_{yy}g_{xyy} - 30g_{xxy}^2g_{xx}g_{yy}g_{xy} - 4g_{xxxy}g_{xy}^2g_{xx}g_{yy} + 4g_{xy}g_{xxxx}g_{yy}^2g_{xx} \\
&+ 12g_{xy}g_{xx}^2g_{xxyy}g_{yy} + 8g_{xx}g_{yyy}g_{xxx}g_{xy}^2 - 12g_{xy}g_{xxx}g_{yy}g_{xx}g_{xyy}), \\
\partial_x\xi_3 = \frac{1}{16\mathsf{d}^{5/4}} (&4g_xg_{xxxx}g_{yy}^2g_{xy}^2 - 4g_xg_{xxxx}g_{yy}^3g_{xx} + 4g_{xx}^2g_yg_{xyyy}g_{xy}^2
\end{aligned}
$$

$$
\begin{aligned}
&- \ 4g_{xx}^3 g_y g_{xyyy} g_{yy} + 12 g_{yy}^2 g_x g_{xxy}^2 g_{xx} - 24 g_x g_{xyy} g_{xy}^3 g_{xxy} + 18 g_x g_{xyy}^2 g_{xy}^2 g_{xx} \\
&- \ 7g_{yy}^2 g_y g_{xxx}^2 g_{xy} + 30 g_{yy} g_x g_{xy}^2 g_{xxy}^2 - 21 g_{xx}^2 g_y g_{xy} g_{xyy}^2 + 3 g_{xx}^2 g_{yy} g_x g_{xyy}^2 \\
&+ \ 7g_{xx}^3 g_y g_{yyy} g_{xyy} - 12 g_{yy} g_x g_{xy}^3 g_{xxxy} - 12 g_{xx} g_y g_{xy}^3 g_{xxyy} - 16 g_{xxx} g_y g_{xy}^3 g_{xyy} \\
&- \ 4g_{xx}^2 g_{yy}^2 g_y g_{xxxy} - 4 g_{xx}^2 g_{yy}^2 g_x g_{xxyy} - 4 g_{yy} g_y g_{xxxx} g_{xy}^3 - 4 g_{xx} g_{xy}^3 g_x g_{xyyy} \\
&- \ 4g_{xxx} g_{xy}^3 g_x g_{yyy} - 14 g_{xx}^2 g_y g_{yyy} g_{xy} g_{xxy} - g_{xx}^2 g_y g_{yyy} g_{xxx} g_{yy} \\
&+ \ 2g_{xx} g_{yy}^2 g_x g_{xyy} g_{xxx} + 15 g_{xx}^2 g_{yy} g_y g_{xxy} g_{xyy} + 7 g_{xx} g_{yy}^2 g_y g_{xxy} g_{xxx} \\
&- \ 30 g_{xx} g_{yy} g_y g_{xxy}^2 g_{xy} + 48 g_{xx} g_y g_{xy}^2 g_{xyy} g_{xxy} + 8 g_{xy}^4 g_y g_{xxxy} \\
&- \ 39 g_{xx} g_{yy} g_x g_{xyy} g_{xy} g_{xxy} - 12 g_{xx} g_y g_{xy} g_{xyy} g_{xxx} g_{yy} - 3 g_{xx} g_{xy} g_x g_{yyy} g_{xxx} g_{yy} \\
&- \ 35 g_x g_{xxx} g_{yy}^2 g_{xy} g_{xxy} + 10 g_{xx} g_{xy}^2 g_x g_{yyy} g_{xxy} - 7 g_{xx}^2 g_{xy} g_x g_{yyy} g_{xyy} \\
&+ \ 28 g_{yy} g_y g_{xxx} g_{xy}^2 g_{xxy} + 7 g_x g_{xxx}^2 g_{yy}^3 + 4 g_{xx}^2 g_{xxy} g_x g_{yyy} g_{yy} \\
&+ \ 4g_{xx}^2 g_{xy} g_x g_{xyyy} g_{yy} + 4 g_{yy}^2 g_y g_{xxxx} g_{xy} g_{xx} + 8 g_{xx} g_y g_{yyy} g_{xxx} g_{xy}^2 \\
&+ \ 12 g_{xx}^2 g_y g_{xy} g_{xxyy} g_{yy} + 12 g_{yy}^2 g_x g_{xy} g_{xxxy} g_{xx} + 26 g_x g_{xyy} g_{xy}^2 g_{xxx} g_{yy} \\
&- \ 12 g_{xy}^3 g_y g_{xxy}^2 + 8 g_x g_{xxyy} g_{xy}^4 - 4 g_{xy}^2 g_y g_{xxxy} g_{xx} g_{yy} - 4 g_x g_{xxyy} g_{xy}^2 g_{xx} g_{yy})
\end{aligned}
$$

e

$$
\begin{aligned}
\partial_y \xi_1 = \frac{1}{16\mathsf{d}^{5/4}} & (-12 g_{xy}^3 g_{xxyy} g_{yy} - 12 g_{xx} g_{xy} g_{yyy} g_{xxy} g_{yy} - 12 g_{xy}^3 g_{xyy}^2 + 8 g_{xy}^4 g_{xyyy} \\
&- \ 7g_{xx}^2 g_{xy} g_{yyy}^2 + 7 g_{xxx} g_{yy}^3 g_{xxy} - 21 g_{xy} g_{xxy}^2 g_{yy}^2 - 4 g_{xx}^2 g_{xyyy} g_{yy}^2 - 16 g_{xxy} g_{xy}^3 g_{yyy} \\
&- \ -4 g_{xx} g_{xy}^3 g_{yyyy} - 4 g_{xxxy} g_{yy}^3 g_{xx} + 4 g_{xxxy} g_{yy}^2 g_{xy}^2 + 15 g_{xx} g_{xyy} g_{yy}^2 g_{xxy} \\
&+ \ 7g_{xx}^2 g_{xyy} g_{yy} g_{yyy} - 30 g_{xx} g_{xyy}^2 g_{yy} g_{xy} + 28 g_{xx} g_{xy}^2 g_{yyy} g_{xyy} - g_{xxx} g_{yy}^2 g_{xx} g_{yyy} \\
&- \ 14 g_{xxx} g_{yy}^2 g_{xy} g_{xyy} + 48 g_{xy}^2 g_{xxy} g_{yy} g_{xyy} - 4 g_{xx} g_{xyyy} g_{yy} g_{xy}^2 \\
&+ \ 4g_{xx}^2 g_{xy} g_{yyyy} g_{yy} + 8 g_{xxx} g_{yy} g_{yyy} g_{xy}^2 + 12 g_{xy} g_{xxyy} g_{yy}^2 g_{xx}), \\
\partial_y \xi_2 = \frac{1}{16\mathsf{d}^{5/4}} & (-4 g_{xy}^3 g_{xxxy} g_{yy} + 12 g_{xx}^2 g_{xyy}^2 g_{yy} - 39 g_{xx} g_{xyy} g_{yy} g_{xy} g_{xxy} \\
&- \ 3g_{xx} g_{xy} g_{yyy} g_{xxx} g_{yy} + 8 g_{xy}^4 g_{xxyy} + 4 g_{xx} g_{xyy} g_{yy}^2 g_{xxx} - 35 g_{xx}^2 g_{xy} g_{yyy} g_{xyy} \\
&+ \ 26 g_{xx} g_{xy}^2 g_{yyy} g_{xxy} - 7 g_{xxx} g_{yy}^2 g_{xy} g_{xxy} + 10 g_{xy}^2 g_{xyy} g_{xxx} g_{yy} - 4 g_{xx} g_{xxyy} g_{yy} g_{xy}^2 \\
&+ \ 2g_{xx}^2 g_{xxy} g_{yyy} g_{yy} + 12 g_{xx}^2 g_{xy} g_{xyyy} g_{yy} + 4 g_{xy} g_{xxxy} g_{yy}^2 g_{xx} + 18 g_{xy}^2 g_{xxy}^2 g_{yy} \\
&+ \ 30 g_{xy}^2 g_{xyy}^2 g_{xx} - 24 g_{xy}^3 g_{xyy} g_{xxy} - 4 g_{xx}^2 g_{xxyy} g_{yy}^2 - 4 g_{xxx} g_{xy}^3 g_{yyy} + 7 g_{xx}^3 g_{yyy}^2 \\
&+ \ 3g_{xxy}^2 g_{yy}^2 g_{xx} - 4 g_{xx}^3 g_{yyyy} g_{yy} + 4 g_{xx}^2 g_{yyyy} g_{xy}^2 - 12 g_{xx} g_{xy}^3 g_{xyyy}), \\
\partial_y \xi_3 = \frac{1}{16\mathsf{d}^{5/4}} & (-21 g_{yy}^2 g_x g_{xy} g_{xxy}^2 + 7 g_x g_{xxx} g_{yy}^3 g_{xxy} - 7 g_{xx}^2 g_{xy} g_x g_{yyy}^2 \\
&+ \ 30 g_{xx} g_y g_{xy}^2 g_{xyy}^2 + 3 g_{xx} g_{yy}^2 g_y g_{xxy}^2 - 12 g_{xx} g_y g_{xy}^3 g_{xyyy} - 4 g_{xx}^2 g_{yy}^2 g_y g_{xxyy} \\
&- \ 4g_{xx}^2 g_{yy}^2 g_x g_{xyyy} - 4 g_{yy} g_y g_{xxxy} g_{xy}^3 - 4 g_{yyy} g_y g_{xxx} g_{xy}^3 - 4 g_{xx} g_{xy}^3 g_x g_{yyyy} \\
&+ \ 4g_x g_{xxxy} g_{yy}^2 g_{xy}^2 - 4 g_x g_{xxxy} g_{yy}^3 g_{xx} + 4 g_{xx}^2 g_y g_{yyyy} g_{xy}^2 - 4 g_{xx}^3 g_y g_{yyyy} g_{yy} \\
&+ \ 12 g_{xx}^2 g_y g_{xyy}^2 g_{yy} - 24 g_{xy}^3 g_y g_{xxy} g_{xyy} + 18 g_{xy}^2 g_y g_{xxy}^2 g_{yy} - 12 g_{yy} g_x g_{xy}^3 g_{xxyy} \\
&- \ 16 g_{xxy} g_{xy}^3 g_x g_{yyy} + 8 g_{xy}^4 g_y g_{xxyy} + 8 g_x g_{xyyy} g_{xy}^4 - 39 g_{xx} g_{yy} g_y g_{xxy} g_{xy} g_{xyy}
\end{aligned}
$$

$$
\begin{aligned}
&- \ 12g_{xx}g_{xy}g_xg_{yyy}g_{xxy}g_{yy} - 3g_{yy}g_yg_{xxx}g_{xy}g_{xx}g_{yyy} - 12g_xg_{xyy}^2g_{xy}^3 \\
&+ \ 2g_{xx}^2g_yg_{yyy}g_{xxy}g_{yy} - 35g_{xx}^2g_yg_{yyy}g_{xy}g_{xyy} + 15g_{xx}g_{yy}^2g_xg_{xyy}g_{xxy} \\
&+ \ 7g_{xx}^2g_{yy}g_xg_{xyy}g_{yyy} - 30g_{xx}g_{yy}g_xg_{xyy}^2g_{xy} + 28g_{xx}g_{xy}^2g_xg_{yyy}g_{xyy} \\
&- \ g_xg_{xxx}g_{yy}^2g_{xx}g_{yyy} - 14g_xg_{xxx}g_{yy}^2g_{xy}g_{xyy} + 48g_{yy}g_xg_{xy}^2g_{xxy}g_{xyy} \\
&- \ 7g_{yy}^2g_yg_{xxx}g_{xy}g_{xxy} + 10g_{yy}g_yg_{xxx}g_{xy}^2g_{xyy} + 26g_{xy}^2g_yg_{xxy}g_{xx}g_{yyy} \\
&- \ 4g_{xy}^2g_yg_{xxyy}g_{xx}g_{yy} - 4g_xg_{xyyy}g_{xy}^2g_{xx}g_{yy} + 4g_{xx}^2g_{xy}g_xg_{yyyy}g_{yy} \\
&+ \ 4g_{yy}^2g_yg_{xxxy}g_{xy}g_{xx} + 4g_{yy}^2g_yg_{xxx}g_{xyy}g_{xx} + 12g_{xx}^2gyg_{xy}g_{xyyy}g_{yy} \\
&+ \ 8g_xg_{xxx}g_{yy}g_{yyy}g_{xy}^2 + 12g_{yy}^2g_xg_{xy}g_{xxyy}g_{xx} + 7g_{xx}^3g_yg_{yyy}^2).
\end{aligned}
$$

Notemos que agora precisamos determinar as derivadas até a quarta ordem de g, o que implica que temos que calcular as derivadas de f também até essa ordem. A seguir, exibiremos apenas uma dessas derivadas de g até a quarta ordem para exemplificar o quanto o cálculo direto é caro.

$$
\begin{aligned}
g_{xxyy} \quad = \quad & -\frac{f_{xxyy}}{f_z} + \frac{f_{xxz}f_{yy} + 2f_{xz}f_{xyy} + 2f_{xxy}f_{yz} + 2f_{xxyz}f_y + 4f_{xy}f_{xyz}}{f_z^2} \\
+ \quad & \frac{f_{xx}f_{yyz} + 2f_xf_{xyyz}}{f_z^2} \\
- \quad & \frac{2f_{xxz}f_{yz}f_y + 2f_{xx}f_{yzz}f_y + f_{xx}f_{zz}f_{yy} + 4f_xf_{xyzz}f_y + f_{xxzz}f_y^2}{f_z^3} \\
- \quad & \frac{2(f_{yz}f_{xx} + 2f_xf_{xyz} + 2f_{xz}f_{xy} + f_{xxz}f_y)f_{yz} + 4f_{xy}f_{xzz}f_y + 2f_{xxy}f_{zz}f_y}{f_z^3} \\
- \quad & \frac{2f_{xz}f_xf_{yyz} - 4(f_{yz}f_x + f_yf_{xz})f_{xyz} - 2f_{zz}f_xf_{xyy} + 2f_{xzz}f_xf_{yy} + f_x^2f_{yyzz}}{f_z^3} \\
- \quad & \frac{2f_{xz}(2f_{yz}f_{xy} + f_xf_{yyz} + f_{xz}f_{yy} + 2f_{xyz}f_y) - 4f_xf_{yzz}f_{xy} - 2f_{zz}f_{xy}^2}{f_z^3} \\
+ \quad & \frac{4f_{zz}(f_{yz}f_x + f_yf_{xz})f_{xy} + 2f_{xz}f_xf_{zz}f_{yy} + 2f_{xx}f_{zz}f_{yz}f_y + 4f_yf_{zz}f_xf_{xyz}}{f_z^4} \\
+ \quad & \frac{2(f_{yz}f_{xx} + 2f_xf_{xyz} + 2f_{xz}f_{xy} + f_{xxz}f_y)f_{zz}f_y + 4f_xf_{yzz}(f_{yz}f_x + f_yf_{xz})}{f_z^4} \\
+ \quad & \frac{f_x^2f_{zzz}f_{yy} + 4f_{xz}f_xf_{yzz}f_y + 2f_x^2f_{yzzz}f_y + f_{xxz}f_{zz}f_y^2}{f_z^4} \\
+ \quad & \frac{2f_{zz}f_x(2f_{yz}f_{xy} + f_xf_{yyz} + f_{xz}f_{yy} + 2f_{xyz}f_y) + 4(f_{yz}f_x + f_yf_{xz})f_{xzz}f_y}{f_z^4} \\
+ \quad & \frac{-2f_{xz}(-4f_{yz}f_yf_{xz} - 2f_{zz}f_yf_{xy} - f_{xzz}f_y^2 - 2f_xf_{yzz}f_y - 2f_{yz}^2f_x - f_{zz}f_xf_{yy})}{f_z^4} \\
+ \quad & \frac{-2(-2f_{xz}^2f_y - f_x^2f_{yzz} - f_{zz}f_yf_{xx} - 2f_{zz}f_xf_{xy} - 4f_{xz}f_{yz}f_x - 2f_xf_{xzz}f_y)f_{yz}}{f_z^4} \\
+ \quad & \frac{f_{zz}f_x^2f_{yyz} + 2f_y^2f_xf_{xzzz} + 4f_{xzz}f_xf_{yz}f_y + f_{xx}f_y^2f_{zzz} + 4f_xf_{zzz}f_yf_{xy}}{f_z^4}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2f_{zz}f_x(-4f_{yz}f_yf_{xz} - 2f_{zz}f_yf_{xy} - f_{xzz}f_y^2 - 2f_xf_{yzz}f_y - 2f_{yz}^2f_x - f_{zz}f_xf_{yy})}{f_z^5}\\
&+ \frac{-6f_{zz}f_x^2f_{yzz}f_y - 2f_x^2f_{zzz}f_{yz}f_y - 6f_y^2f_{zz}f_xf_{xzz}}{f_z^5}\\
&- \frac{2f_{zz}^2(2f_yf_{zz}f_xf_{xy} + f_{yz}f_x + f_yf_{xz}) - 2(f_x^2f_{zzz}f_y + 6f_{xz}f_yf_{zz}f_x + 3f_{zz}f_x^2f_{yz})f_{yz}}{f_z^5}\\
&+ \frac{-2f_{xz}f_xf_y^2f_{zzz} - f_{zz}^2f_x^2f_{yy} - 4f_{xz}f_xf_{zz}f_{yz}f_y - f_{xx}f_{zz}^2f_y^2}{f_z^5}\\
&+ \frac{-4f_xf_{zzz}f_y(f_{yz}f_x + f_yf_{xz}) - 2f_{xz}(3f_{xz}f_{zz}f_y^2 + 6f_{yz}f_yf_{zz}f_x + f_y^2f_xf_{zzz})}{f_z^5}\\
&+ \frac{2(-2f_{xz}^2f_y - f_x^2f_{yzz} - f_{zz}f_yf_{xx} - 2f_{zz}f_xf_{xy} - 4f_{xz}f_{yz}f_x - 2f_xf_{xzz}f_y)f_{zz}f_y}{f_z^5}\\
&+ \frac{-f_x^2f_y^2f_{zzzz}}{f_z^5}\\
&+ \frac{6f_x^2f_{zzz}f_y^2f_{zz} + 8f_{xz}f_{zz}^2f_xf_y^2 + 8f_{zz}^2f_x^2f_yf_{yz}}{f_z^6}\\
&+ \frac{4f_{zz}^2f_yf_x(f_{yz}f_x + f_yf_{xz}) + 2f_{zz}f_x(3f_{xz}f_{zz}f_y^2 + 6f_{yz}f_yf_{zz}f_x + f_y^2f_xf_{zzz})}{f_z^6}\\
&+ \frac{2(f_x^2f_{zzz}f_y + 6f_{xz}f_yf_{zz}f_x + 3f_{zz}f_x^2f_{yz})f_{zz}f_y}{f_z^6} - \frac{15f_y^2f_{zz}^3f_x^2}{f_z^7}.
\end{aligned}
$$

E por fim, resta calcular os coeficientes $(b_{i,j})_{1\leq i,j\leq 2}$ do operador forma $\mathcal{S}$ definidos no capítulo 2

$$
\begin{aligned}
b_{11} &= 12g_{xy}^3g_{xxxy}g_{yy} - 3g_{xx}^2g_{xyy}^2g_{yy} + 39g_{xx}g_{xyy}g_{yy}g_{xy}g_{xxy} + 3g_{xx}g_{xy}g_{yyy}g_{xxx}g_{yy}\\
&- 7g_{xxx}^2g_{yy}^3 - 8g_{xy}^4g_{xxyy} - 2g_{xx}g_{xyy}g_{yy}^2g_{xxx} + 7g_{xx}^2g_{xy}g_{yyy}g_{xyy} - 10g_{xx}g_{xy}^2g_{yyy}g_{xxy}\\
&+ 35g_{xxx}g_{yy}^2g_{xy}g_{xxy} - 26g_{xy}^2g_{xyy}g_{xxx}g_{yy} + 4g_{xx}g_{xxyy}g_{yy}g_{xy}^2 - 4g_{xx}^2g_{xxy}g_{yyy}g_{yy}\\
&- 4g_{xx}^2g_{xy}g_{xyyy}g_{yy} - 12g_{xy}g_{xxxy}g_{yy}^2g_{xx} - 30g_{xy}^2g_{xxy}^2g_{yy} - 18g_{xy}^2g_{xyy}^2g_{xx}\\
&+ 24g_{xy}^3g_{xyy}g_{xxy} + 4g_{xx}^2g_{xxyy}g_{yy}^2 + 4g_{xxx}g_{xy}^3g_{yyy} + 4g_{xx}g_{xy}^3g_{xyyy}\\
&+ 4g_{xxxx}g_{yy}^3g_{xx} - 4g_{xxxx}g_{yy}^2g_{xy}^2 - 12g_{xxy}^2g_{yy}^2g_{xx},\\
b_{12} &= 7g_{xy}g_{xxx}^2g_{yy}^2 + 12g_{xy}^3g_{xx}g_{xxyy} + 12g_{xxy}^2g_{xy}^3 - 8g_{xxxy}g_{xy}^4 + 21g_{xy}g_{xx}^2g_{xyy}^2\\
&- 7g_{xx}^3g_{yyy}g_{xyy} + 4g_{xy}^3g_{xxxx}g_{yy} + 16g_{xy}^3g_{xxx}g_{xyy} + 4g_{xx}^3g_{xyyy}g_{yy} - 4g_{xx}^2g_{xyyy}g_{xy}^2\\
&+ 4g_{xxxy}g_{xx}^2g_{yy}^2 - 28g_{xxy}g_{xy}^2g_{xxx}g_{yy} - 48g_{xxy}g_{xy}^2g_{xx}g_{xyy} + g_{xx}^2g_{yyy}g_{xxx}g_{yy}\\
&+ 14g_{xx}^2g_{yyy}g_{xy}g_{xxy} - 7g_{xxy}g_{xx}g_{yy}^2g_{xxx} - 15g_{xxy}g_{xx}^2g_{yy}g_{xyy} + 30g_{xxy}^2g_{xx}g_{yy}g_{xy}\\
&+ 4g_{xxxy}g_{xy}^2g_{xx}g_{yy} - 4g_{xy}g_{xxxx}g_{yy}^2g_{xx} - 12g_{xy}g_{xx}^2g_{xxyy}g_{yy}\\
&- 8g_{xx}g_{yyy}g_{xxx}g_{xy}^2 + 12g_{xy}g_{xxx}g_{yy}g_{xx}g_{xyy},\\
b_{21} &= 12g_{xy}^3g_{xxyy}g_{yy} + 12g_{xx}g_{xy}g_{yyyy}g_{xxy}g_{yy} + 12g_{xy}^3g_{xyy}^2 - 8g_{xy}^4g_{xyyy} + 7g_{xx}^2g_{xy}g_{yyy}^2\\
&- 7g_{xxx}g_{yy}^3g_{xxy} + 21g_{xy}g_{xxy}^2g_{yy}^2 + 4g_{xx}^2g_{xyyy}g_{yy}^2 + 16g_{xxy}g_{xy}^3g_{yyy} + 4g_{xx}g_{xy}^3g_{yyyy}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 4g_{xxxy}g_{yy}^3g_{xx} - 4g_{xxxy}g_{yy}^2g_{xy}^2 - 15g_{xx}g_{xyy}g_{yy}^2g_{xxy} - 7g_{xx}^2g_{xyy}g_{yy}g_{yyy} \\
&+ \quad 30g_{xx}g_{xyy}^2g_{yy}g_{xy} - 28g_{xx}g_{xy}^2g_{yyy}g_{xyy} + g_{xxx}g_{yy}^2g_{xx}g_{yyy} + 14g_{xxx}g_{yy}^2g_{xy}g_{xyy} \\
&- \quad 48g_{xy}^2g_{xxy}g_{yy}g_{xyy} + 4g_{xx}g_{xyyy}g_{yy}g_{xy}^2 - 4g_{xx}^2g_{xy}g_{yyyy}g_{yy} \\
&- \quad 8g_{xxx}g_{yy}g_{yyy}g_{xy}^2 - 12g_{xy}g_{xxyy}g_{yy}^2g_{xx}
\end{aligned}
$$

e

$$
\begin{aligned}
b_{22} \quad = \quad & 4g_{xy}^3g_{xxxy}g_{yy} - 12g_{xx}^2g_{xyy}^2g_{yy} + 39g_{xx}g_{xyy}g_{yy}g_{xy}g_{xxy} + 3g_{xx}g_{xy}g_{yyy}g_{xxx}g_{yy} \\
- \quad & 8g_{xy}^4g_{xxyy} - 4g_{xx}g_{xyy}g_{yy}^2g_{xxx} + 35g_{xx}^2g_{xy}g_{yyy}g_{xyy} - 26g_{xx}g_{xy}^2g_{yyy}g_{xxy} \\
+ \quad & 7g_{xxx}g_{yy}^2g_{xy}g_{xxy} - 10g_{xy}^2g_{xyy}g_{xxx}g_{yy} + 4g_{xx}g_{xxyy}g_{yy}g_{xy}^2 - 2g_{xx}^2g_{xxy}g_{yyy}g_{yy} \\
- \quad & 12g_{xx}^2g_{xy}g_{xyyy}g_{yy} - 4g_{xy}g_{xxxy}g_{yy}^2g_{xx} - 18g_{xy}^2g_{xxy}^2g_{yy} - 30g_{xy}^2g_{xyy}^2g_{xx} \\
+ \quad & 24g_{xy}^3g_{xyy}g_{xxy} + 4g_{xx}^2g_{xxyy}g_{yy}^2 + 4g_{xxx}g_{xy}^3g_{yyy} + 12g_{xx}g_{xy}^3g_{xyyy} - 3g_{xxy}^2g_{yy}^2g_{xx} \\
- \quad & 7g_{xx}^3g_{yyy}^2 + 4g_{xx}^3g_{yyyy}g_{yy} - 4g_{xx}^2g_{yyyy}g_{xy}^2.
\end{aligned}
$$

Observemos que ao trabalharmos com o *método direto* temos um enorme número de derivadas e isso acarreta erros numéricos. O que pode ser visto claramente nas expressões do cálculo dos coeficientes b_{ij} antes e depois da transformação (seção 3.3.2).