4
The simplified-hybrid boundary element method for axisymmetric elasticity

The simplified-hybrid boundary element method [51, 52] was proposed about one decade ago as a simplified version of the variationally-based, hybrid boundary element method [42, 43]. However, this formulation can be derived independently on the basis of a virtual work statement and a nodal displacement compatibility equation, which is the approach adopted in the following chapter.

Since its original formulation, the simplified-hybrid boundary element method has undergone some slight, although relevant, conceptual improvements, which helped to elucidate the theoretical aspects of the method and increase the range of applications. However, some topological features that are inherent to axisymmetric problems have been at first seen as untractable in the proposed variational framework.

This chapter basically presents a new theoretical development that has made possible the application of the simplified-hybrid boundary element method to axisymmetric problems [60], as well. New improvements provide the definitive answer to some conceptual issues that had been identified, but not solved, since the onset of the method. The key contribution is the development of a hybrid virtual work statement to arrive at a contragradient theorem that opens up the possibility of application of the method to problems of any topology.

In the next sections, one firstly presents the simplified-hybrid boundary element method, as originally proposed [51, 52] and particularized to axisymmetric problems, showing why some undetermined coefficients cannot be resolved for some geometry configurations. Then, one introduces the above-mentioned hybrid virtual work statement, that subtly, but definitely, turns the method completely applicable.

4.1
Formulation for the axisymmetric fullspace problem

4.1.1
Approximation of displacements and tractions

Let an axisymmetric body be submitted to body forces b_i in Ω, traction forces $\bar{t}_i$ on Γ_σ and prescribed displacements $\bar{u}_i$ on Γ_u, as depicted in Fig. 4.1. One is looking for an approximation of displacements u_i along Γ_σ and of traction forces t_i along Γ_u that best satisfy the equilibrium equations in the domain Ω. In the simplified-hybrid boundary element method, the displacement and stress fields are approximated by two independent trial fields, namely displacements u_i on the boundary Γ and stresses σ^*_{ij} in the domain Ω, according to the methodology proposed by Pian [44].

Then, displacements are approximated along the boundary Γ by

$$u_i = u_{in}\, u_n \qquad \text{where} \quad n = 1, n_u \tag{4-1}$$

in which u_n are nodal values and u_{in} are interpolation functions with compact support. The boundary is discretized in terms of n_e segments Γ_e with a total of n_n nodes, in the frame of an isoparametric formulation, as in the conventional boundary element method.

In the domain, the stress field is approximated by a series of fundamental solutions – as homogeneous solutions of the equilibrium differential equation of the problem – plus some arbitrary particular solution that takes the applied body forces int account. In the present case, the fundamental solutions are obtained by applying ring sources of intensity p^*_m at each node and coordinate direction, as already outlined in Section 3.1.1. Then, the homogeneous part of the stresses at any point $\mathrm{Q}(r, z)$ in the domain Ω is given by

$$\sigma^*_{ij}(\mathrm{Q}) = \sigma^{*\mathrm{f}}_{ijm}(\mathrm{Q}, \mathrm{P})\, p^*_m(\mathrm{P}) \tag{4-2}$$

where $\sigma^{*\mathrm{f}}_{ijm}$ is the fullspace fundamental solution. The index m indicates both the point $\mathrm{P}(\xi, z')$ and the direction at which the load is applied; the subscripts ij refer to the components stresses, as measured at $\mathrm{Q}(r, z)$. One also obtains from the stress field a displacement field whose homogeneous part is

$$u^*_i(\mathrm{Q}) = u^{*\mathrm{f}}_{im}(\mathrm{Q}, \mathrm{P})\, p^*_m(\mathrm{P}) + c^r_i \tag{4-3}$$

where $u^{*\mathrm{f}}_{im}$ is the corresponding displacement fundamental solution for the fullspace and and c^r_i are in principle arbitrary rigid body constants. For axisymmetric problems, the only possible rigid body motion is translation in the z-direction, formally given by

$$c^r_i = u^r_i(\mathrm{Q})\, C_m(\mathrm{P})\, p^*_m(\mathrm{P}) \tag{4-4}$$

where C_m are in principle arbitrary constants and u^r_i are the rigid body displacements

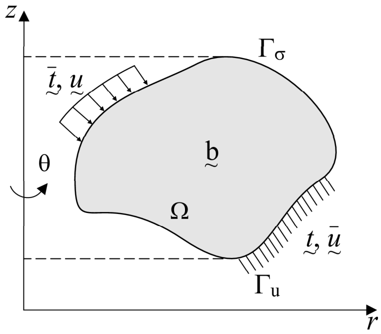
$$u^r_i = \begin{pmatrix} 0 \\ n_n^{-1/2} \end{pmatrix} \tag{4-5}$$

where n_n is the number of nodes. These interpolation functions are normalized in such a way that, if one takes $Q(r, z)$ to the boundary, their values coincide with the orthonormal basis of rigid body displacements $\mathbf{W} = [W_m] \in \mathbb{R}^{n_u}$

$$W_m = \begin{cases} 0 & \text{if } m \text{ refers to } r\text{-direction} \\ n_n^{-1/2} & \text{if } m \text{ refers to } z\text{-direction} \end{cases} \tag{4-6}$$

The above definition is adopted for the sake of simplicity, as $\mathbf{W}\mathbf{W}^{\mathrm{T}}$ is an orthogonal projector.

The rigid body displacement constants C_m – in principle arbitrary – are evaluated in due time, as shown in two different circumstances in Sections 4.1.2.2 and 4.1.4.1.



(a)

Figure 4.1: Meridian plane of an elastic axisymmetric body submitted to body forces, traction forces and prescribed displacements

4.1.2
Governing matrix equation

4.1.2.1
Displacement virtual work for nodal equilibrium checking

One states the following displacement virtual work principle to check equilibrium of forces in the elastic body illustrated in Fig. 4.1:

$$\int_\Omega \underset{\approx}{\sigma}^* : \delta\underset{\approx}{\epsilon}\, \mathrm{d}\Omega = \int_\Omega \underset{\sim}{b} \cdot \delta\underset{\sim}{u}\, \mathrm{d}\Omega + \int_\Gamma \underset{\sim}{t} \cdot \delta\underset{\sim}{u}\, \mathrm{d}\Gamma \tag{4-7}$$

where $\delta\underset{\sim}{u}$ are virtual displacements such that $\delta\underset{\sim}{u} = 0$ along Γ_u. Assuming that the stress tensor $\underset{\approx}{\sigma}^*$ is symmetric and considering Eq. (2-2), one may rewrite the first integral above as

$$\int_\Omega \underset{\approx}{\sigma}^* : \delta\underset{\approx}{\epsilon}\, \mathrm{d}\Omega = \int_\Omega \underset{\approx}{\sigma}^* : (\underset{\sim}{\nabla}\delta\underset{\sim}{u})\, \mathrm{d}\Omega \tag{4-8}$$

which, after integration by parts, leads to

$$\int_\Omega \underset{\approx}{\sigma}^* : \delta\underset{\approx}{\epsilon}\, \mathrm{d}\Omega = \int_\Omega \underset{\sim}{\nabla} \cdot (\underset{\approx}{\sigma}^* \cdot \delta\underset{\sim}{u})\, \mathrm{d}\Omega - \int_\Omega \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}^* \cdot \delta\underset{\sim}{u}\, \mathrm{d}\Omega \tag{4-9}$$

Applying the divergence theorem and substituting for Eq. (2-19), the first integral in Eq. (4-7) yields

$$\int_\Omega \underset{\approx}{\sigma}^* : \delta\underset{\approx}{\epsilon}\, \mathrm{d}\Omega = \int_\Gamma \underset{\sim}{t}^* \cdot \delta\underset{\sim}{u}\, \mathrm{d}\Gamma - \int_\Omega \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}^* \cdot \delta\underset{\sim}{u}\, \mathrm{d}\Omega \tag{4-10}$$

For the sake of simplicity, one shall assume in the following developments that body forces are absent. The above relation can be substituted into Eq. (4-7), which becomes

$$\int_\Gamma \underset{\sim}{t}^* \cdot \delta\underset{\sim}{u}\, \mathrm{d}\Gamma - \int_\Omega \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}^* \cdot \delta\underset{\sim}{u}\, \mathrm{d}\Omega = \int_\Gamma \underset{\sim}{t} \cdot \delta\underset{\sim}{u}\, \mathrm{d}\Gamma \tag{4-11}$$

Approximating the stresses $\underset{\approx}{\sigma}^*$ in the domain according to Eq. (4-2), as well as the displacements u_i on the boundary according to Eq. (4-1), and applying the property of the fundamental solution given by Eq. (2-77), one obtains, in indicial notation,

$$\delta u_n \left(\int_\Gamma t^{*\mathrm{f}}_{im}(\mathrm{Q},\mathrm{P})\, u_{in}(\mathrm{Q})\, \mathrm{d}\Gamma + \delta^*_{im}\, u_{in}(\mathrm{Q}) \right) p^*_m(\mathrm{P}) = \delta u_n \int_\Gamma t_i(\mathrm{Q})\, u_{in}(\mathrm{Q})\, \mathrm{d}\Gamma \tag{4-12}$$

One identifies the last integral as the expression of equivalent nodal forces, as already given in Eq. (3-18), thus arriving at

$$\delta u_n \left(\int_\Gamma t^{*\mathrm{f}}_{im}(\mathrm{Q},\mathrm{P})\, u_{in}(\mathrm{Q})\, \mathrm{d}\Gamma + \delta^*_{im}\, u_{in}(\mathrm{Q}) \right) p^*_m(\mathrm{P}) = \delta u_n\, p_n \tag{4-13}$$

Considering $\delta^*_{im}\, u_{in}(\mathrm{Q}) = u_{mn}(\mathrm{Q}) = \delta^*_{mn}$ and the boundary $\mathrm{d}\Gamma$ given as in Eq. (3-5), one obtains for any δu_n and after integration over θ,

$$H_{mn}\, p^*_m = p_n \quad \text{or} \quad \mathbf{H}^{\mathrm{T}}\, \mathbf{p}^* = \mathbf{p} \tag{4-14}$$

in which

$$H_{mn} = 2\pi \int_\Gamma t^{*\mathrm{f}}_{im}(\mathrm{Q},\mathrm{P})\, u_{in}(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) + \delta^*_{mn} = \widehat{H}_{mn} + c^{\mathrm{c}}_{mn} \tag{4-15}$$

in terms of the Cauchy principal value of the singular integral

$$\widehat{H}_{mn} = 2\pi \fint_\Gamma t^{*\mathrm{f}}_{im}(\mathrm{Q},\mathrm{P})\, u_{in}(\mathrm{Q})\, r\, \mathrm{d}\Gamma(r,z) \tag{4-16}$$

The discontinuous part of the singular integral in Eq. (4-14) is included in the constants c^{c}_{mn}, as outlined in Section 3.1.5. Then, the matrix $\mathbf{H} = [H_{mn}] \in \mathbb{R}^{n_u \times n_u}$

expressed in Eq. (4-15) is the same one of the boundary element method, introduced in the previous chapter. In Eq. (4-14), the matrix $\mathbf{H}^T$ transforms forces $\mathbf{p}^* = [p^*_m] \in \mathbb{R}^{n_u}$ of the auxiliary system into the nodal equivalent forces $\mathbf{p} = [p_n] \in \mathbb{R}^{n_u}$.

The *external* reference system $(\mathbf{u}, \mathbf{p})$, with nodal displacements $\mathbf{u}$ introduced in Eq. (4-1) and equivalent nodal forces $\mathbf{p}$ obtained in the frame of the virtual work statement that has led to Eq. (4-14), provides the numerical approximation of the actions along the boundary of the elastic body. The *internal* reference system $(\mathbf{u}^*, \mathbf{p}^*)$ approximates stresses in the domain in terms of point forces $\mathbf{p}^*$, to which correspond equivalent nodal displacements $\mathbf{u}^*$, defined in the following contragradient statement.

In fact, one may write the virtual work statement

$$\delta\mathbf{p}^{*T}\mathbf{u}^* = \delta\mathbf{p}^T\mathbf{u} \tag{4-17}$$

for virtual forces $\delta\mathbf{p}^*$ and $\delta\mathbf{p}$ that are in equilibrium. Substituting for $\delta\mathbf{p}$ according to Eq. (4-14), one obtains

$$\mathbf{H}\,\mathbf{u} = \mathbf{u}^* \tag{4-18}$$

This equation is also integrant part of the variationally-based hybrid boundary element method and might be inferred in the frame of the conventional boundary element method [43].

4.1.2.2
Nodal displacement compatibility

Equation (4-3), which approximates the displacements u^f_i in the domain Ω, is also valid at the boundary nodal points, as obtained in the frame of the variational hybrid boundary element method [43]. Therefore, applying Eqs. (4-3) and (4-4) to each node leads to

$$u_n = (U^*_{nm} + W_n\,C_m)\,p^*_m \quad \text{or} \quad \mathbf{u} = (\mathbf{U}^* + \mathbf{W}\,\mathbf{C})\,\mathbf{p}^* \tag{4-19}$$

where $\mathbf{W} = [W_n] \in \mathbb{R}^{n_u}$ is the orthonormal basis of the rigid body displacements given by Eq. (4-6) and $\mathbf{C} = [C_m] \in \mathbb{R}^{n_u}$ are constants to be obtained in the following steps. In the above expression, the interpolation functions u^r_i are normalized according to Eq. (4-5). The index m refers both to the point $\mathrm{P}(\xi, z')$ at which the load is applied and to the application direction, while the index n refers both to the point $\mathrm{Q}(r, z)$ at which the displacement is evaluated and to the coordinate direction.

In Eq. (4-19), $\mathbf{U}^* = [U^*_{nm}] \in \mathbb{R}^{n_u \times n_u}$ is a matrix with coefficients given as the displacements $u^{*f}_{im}(\mathrm{Q}, \mathrm{P})$ measured at the degrees of freedom n. The displacements

$u^{*\mathrm{f}}_{im}(\mathrm{Q},\mathrm{P})$ tend to infinity when m and n refer to the same node. In such a case, however, the corresponding coefficients of $\mathbf{U}^*$, if expressed from Eq. (4-3), are actually undetermined, as they would refer to a point that is outside the domain of interest Ω, although at only an infinitesimal distance. Developing a generally valid procedure to evaluate such coefficients of $\mathbf{U}^*$ is the key issue of the hybrid boundary element methods (no matter which version one is using [43]), as dealt in Section 4.1.4. Regardless the fact that these coefficients have not yet been evaluated, one may proceed with the present developments. By construction, $U^*_{nm} = U^*_{mn}$ when m and n refer to different nodes. However, no symmetry properties can in principle – and also ultimately – be derived for the submatrices about the main diagonal of $\mathbf{U}^*$.

Pre-multiplying Eq. (4-19) by $\mathbf{W}^{\mathrm{T}}$ and using the property $\mathbf{W}^{\mathrm{T}}\mathbf{W} = 1$, one solves for the product $\mathbf{C}\,\mathbf{p}^*$ in a least-squares sense:

$$\mathbf{C}\,\mathbf{p}^* = \mathbf{W}^{\mathrm{T}}(\mathbf{u} - \mathbf{U}^*\,\mathbf{p}^*) \tag{4-20}$$

Substituting the above expression into Eq. (4-19) leads to

$$\mathbf{P}^{\perp}_{\mathrm{W}}\,\mathbf{U}^*\,\mathbf{p}^* = \mathbf{P}^{\perp}_{\mathrm{W}}\,\mathbf{u} \tag{4-21}$$

where

$$\mathbf{P}^{\perp}_{\mathrm{W}} = \mathbf{I} - \mathbf{P}_{\mathrm{W}} \tag{4-22}$$

$$\mathbf{P}_{\mathrm{W}} = \mathbf{W}\,\mathbf{W}^{\mathrm{T}} \tag{4-23}$$

in which $\mathbf{P}_{\mathrm{W}}$ is the orthogonal projector that spans the space of rigid body displacements and $\mathbf{P}^{\perp}_{\mathrm{W}}$ its complementary projector. The matrix $\mathbf{P}^{\perp}_{\mathrm{W}}\,\mathbf{U}^*$ in Eq. (4-21) is singular and transforms forces of the auxiliary system $\mathbf{p}^*$ into displacements $\mathbf{u}$.

4.1.3 Properties of the orthogonal bases W, V, A

Equations (4-14) and (4-21) transform forces and displacements, respectively, between two different approximation reference systems, as proposed in Section 4.1.1.

For axisymmetric problems in the fullspace, there is a basis $\mathbf{W} = [W_m] \in \mathbb{R}^{n_u}$ of rigid body displacements in the z-direction, whose orthogonal projectors $\mathbf{P}_{\mathrm{W}}$ and $\mathbf{P}^{\perp}_{\mathrm{W}}$ are expressed in Eqs. (4-23) and (4-22), respectively. The basis $\mathbf{W}$ spans the nullspace of $\mathbf{H}$, since rigid body displacements are not transformed by Eq. (4-18), i.e.,

$$\mathbf{H}\,\mathbf{W} = \mathbf{0} \tag{4-24}$$

This orthogonal basis depends only on the geometry of the problem, as expressed by Eq. (4-6). The orthogonality relation in the equation above can be used to circumvent the numerical evaluation of the singular integrals of $\mathbf{H}$, for coefficients about the main diagonal, as presented in Chapter 5.

On the other hand, one obtains from Eq. (4-24) that there is a basis $\mathbf{V} = [V_m] \in \mathbb{R}^{u_u}$ that spans the nullspace of $\mathbf{H}^{\mathrm{T}}$,

$$\mathbf{H}^{\mathrm{T}}\,\mathbf{V} = \mathbf{0} \tag{4-25}$$

which is also the basis of inadmissible, unbalanced forces $\mathbf{p}^*$ that cannot be transformed in Eq. (4-14). The corresponding orthogonal projector and complementary orthogonal projector are, for $\mathbf{V}$ used as an orthonormal matrix,

$$\mathbf{P}_{\mathrm{V}} = \mathbf{V}\,\mathbf{V}^{\mathrm{T}} \tag{4-26}$$

$$\mathbf{P}_{\mathrm{V}}^{\perp} = \mathbf{I} - \mathbf{P}_{\mathrm{V}} \tag{4-27}$$

Both $\mathbf{p}^*$ and $\mathbf{p}$ are forces that act in the same direction at the same nodal points and can perform virtual work on the same set of nodal displacements. As a consequence, $\tilde{\mathbf{V}}^{\mathrm{T}}\mathbf{W}$, as a non-normalized basis $\tilde{\mathbf{V}}$ of unbalanced forces $\mathbf{p}^*$, can be projected onto the space spanned by $\mathbf{W}$ of unbalanced forces $\mathbf{p}$:

$$\mathbf{P}_{\mathrm{W}}\,\tilde{\mathbf{V}} = \mathbf{W} \tag{4-28}$$

This equation leads together with Eq. (4-25) to

$$(\mathbf{H}^{\mathrm{T}} + \mathbf{P}_{\mathrm{W}})\,\tilde{\mathbf{V}} = \mathbf{W} \tag{4-29}$$

where $(\mathbf{H}^{\mathrm{T}} + \mathbf{P}_{\mathrm{W}})$ is by definition non-singular. Equation above is an expedite linear algebra means to find the nullspace of $\mathbf{H}^{\mathrm{T}}$.

Equations (4-24) and (4-25) are consistent, since

$$\mathbf{W}^{\mathrm{T}}\,\mathbf{p} = 0 \tag{4-30}$$

$$\mathbf{V}^{\mathrm{T}}\,\mathbf{u}^* = 0 \tag{4-31}$$

Spectral relations of $\mathbf{U}^*$ can be derived from the mechanical interpretation that unbalanced forces spanned by the bases $\mathbf{V}$ and $\mathbf{W}$ cannot reproduce displacements in Eqs. (4-21) and (4-59),

$$\mathbf{P}_{\mathrm{W}}^{\perp}\,\mathbf{U}^{*}\,\mathbf{V} = \mathbf{0} \tag{4-32}$$

$$(\mathbf{P}_{\mathrm{W}}^{\perp}\mathbf{U}^{*})^{\mathrm{T}}\,\mathbf{W} = \mathbf{0} \tag{4-33}$$

Equation (4-32) might be sought as a means to evaluate the elements about the main diagonal of $\mathbf{U}^*$. In fact, this equation is a necessary condition for $\mathbf{U}^*$, whenever the elastic body can undergo rigid body displacements, but a not sufficient one, as $\mathbf{P}_{\mathrm{W}}^{\perp}\,\mathbf{U}^*$ is singular by construction. The spectral condition represented by Eq. (4-32) will be proven consistent with the developments of Section 4.1.5.

For axisymmetric problems, another important basis is $\mathbf{A} = [A_{im}] \in \mathbb{R}^{n_a \times n_u}$, similar to the one introduced in Section 3.1.4.1 but this time redefined for n_u columns, that reflects the fact that radial loads on the axis of axisymmetry generate no displacements. Let n_a be the number of nodes along the boundary on the axis of axisymmetry. For each node i at which $\xi = 0$ one expresses the orthonormal basis $\mathbf{A}$ as

$$A_{im} = \begin{cases} 1 & \text{if } m \text{ refers to node } i \text{ and the } r\text{-direction} \\ 0 & \text{otherwise} \end{cases} \tag{4-34}$$

The orthogonal projector $\mathbf{P}_{\mathrm{A}}$ of the space of the radial loads on the axis of axisymmetry and its complementary projector $\mathbf{P}_{\mathrm{A}}^{\perp}$ are

$$\mathbf{P}_{\mathrm{A}} = \mathbf{A}\,\mathbf{A}^{\mathrm{T}} \tag{4-35}$$

$$\mathbf{P}_{\mathrm{A}}^{\perp} = \mathbf{I} - \mathbf{P}_{\mathrm{A}} \tag{4-36}$$

The basis $\mathbf{A}$ describes the subset of radial loads that are applied on the axis of axisymmetry and that cannot generate any displacements (except for axial rigid body displacements). It equally represents point forces $\mathbf{p}^*$ and equivalent nodal forces $\mathbf{p}$. This can be checked for the equilibrium Eq. (4-14):

$$\mathbf{H}^{\mathrm{T}}\,\mathbf{A} = \mathbf{A} \tag{4-37}$$

Also, since $\mathbf{W}$ refers to the axial direction and $\mathbf{A}$ to the radial direction,

$$\mathbf{W}^{\mathrm{T}}\,\mathbf{A} = 0 \tag{4-38}$$

Since the basis $\mathbf{A}$ of radial loads on the axis of axisymmetry generates no displacements in Eq. (4-21), and considering Eq. (4-38), one obtains

$$\mathbf{U}^{*}\,\mathbf{A} = \mathbf{0} \tag{4-39}$$

Understanding the mechanical meaning of the spectral spaces spanned by **W**, **V** and **A** is a key step in the conceptual completion of the simplified-hybrid boundary element method, including the evaluation of the coefficients about the main diagonal of $\mathbf{U}^*$ and the manipulation of some generalized inverse matrices.

The next section outlines how the coefficients about the main diagonal of $\mathbf{U}^*$ can be evaluated, for the axisymmetric fullspace problem, using the procedure originally proposed in the simplified-hybrid boundary element method [50]. As it is shown, there are some topological problems that cannot be overcome in the frame of the original proposition. On the other hand, Section 4.1.5 introduces a novel formulation that renders the simplified-hybrid boundary element method applicable to any kind of geometry configuration, which is one of the major contributions of the present work.

4.1.4
Evaluation of the coefficients about the main diagonal of $\mathbf{U}^*$ as in the original proposition of the simplified-hybrid boundary element method

As mentioned in Section 4.1.5.1, $\mathbf{U}^* = [U^*_{nm}] \in \mathbb{R}^{u_u \times u_u}$ is a matrix obtained by measuring displacements $u^{*c}_{im}(\mathrm{Q},\mathrm{P})$ at the degrees of freedom n. When m and n refer to the same node, i.e. the points $\mathrm{P}(\xi, z')$ and $\mathrm{Q}(r, z)$ coincide ($\rho = 0$), the functions $u^{*c}_{im}(\mathrm{Q},\mathrm{P})$ is actually not infinity, but indeterminate, as they would refer to a point that is outside the domain of interest Ω. Therefore, a series of two by two submatrices along the main diagonal of U^*_{nm} must be evaluated by some mechanically-based linear algebra means. The following outline is based on the fact that inadmissible forces $\mathbf{p}^*$, thus spanned by the basis **V**, cannot generate an equilibrated stress state in the body.

4.1.4.1
Evaluation of rigid body displacement constants comprised in a plain state of deformation

The rigid body displacement constants C_m introduced in Eq. (4-3) can be evaluated, for a static problem, by imposing that only a plain deformation state be represented by this equation. It means that the total displacements $u^*_i(Q)$, as caused by $p^*_m(P)$, should be orthogonal to any rigid body displacement amount $\tilde{c}_i = u^r_i \tilde{C}_m p^*_m$ measured either in the whole domain or along the boundary [58]. In the following developments, one imposes as orthogonality criterion that

$$\int_\Gamma u^*_i \tilde{c}_i \, \mathrm{d}\Gamma = 2\pi \int_\Gamma u^*_i \tilde{c}_i \, r \, \mathrm{d}\Gamma(r,z) = 0 \qquad (4\text{-}40)$$

in which $\Gamma(r, z)$ is the boundary of the meridian plane of the axisymmetric body. Substituting for u_i^* according to Eqs. (4-3) and (4-4), one obtains for any p_m^*

$$\int_\Gamma (u_{im}^{*\mathrm{f}} + u_i^r\, C_m)\, u_i^r\, r\, \mathrm{d}\Gamma(r, z) = 0 \tag{4-41}$$

Defining $\mathbf{C}^* = [C_m^*] \in \mathbb{R}^{n_u}$ and the scalar C^r as

$$C_m^* = \int_\Gamma u_{im}^{*\mathrm{f}}\, u_i^r\, r\, \mathrm{d}\Gamma(r, z) = n_n^{-1/2} \int_\Gamma u_{zm}^{*\mathrm{f}}\, r\, \mathrm{d}\Gamma(r, z) \tag{4-42}$$

$$C^r = \int_\Gamma u_i^r\, u_i^r\, r\, \mathrm{d}\Gamma(r, z) = n_n^{-1}\, \ell_\Gamma \tag{4-43}$$

one solves for $\mathbf{C} = [C_m] \in \mathbb{R}^{n_u}$ as

$$\mathbf{C} = -\frac{\mathbf{C}^*}{C^r} \tag{4-44}$$

where $2\pi\, \ell_\Gamma = \int_\Gamma r\, \mathrm{d}\Gamma(r, z)$ is the surface of the axisymmetric body. Equation (4-5) and Eqs. (4-42) to (4-44) can be substituted into Eq. (4-4) to derive the amount of rigid body displacements they must be included in Eq. (4-3) so that only displacements related to a plain deformation state are induced by $\mathbf{p}^*$:

$$c_i^r = \begin{pmatrix} 0 \\ -\frac{p_m^*}{\ell_\Gamma} \int_\Gamma u_{zm}^{*\mathrm{f}}\, r\, \mathrm{d}\Gamma(r, z) \end{pmatrix} \tag{4-45}$$

4.1.4.2
A nodal displacement compatibility statement

Since unbalanced forces of the auxiliary system $\mathbf{p}^*$, thus spanned by the basis $\mathbf{V}$, should not reproduce any admissible state of displacements in Eq. (4-19),

$$(\mathbf{U}^* + \mathbf{W}\,\mathbf{C})\,\mathbf{V} = \mathbf{0} \tag{4-46}$$

for the matrix $\mathbf{C}$ of rigid body displacements evaluated according to Eq. (4-44). Let $\mathbf{U}^*$ be split into two parts,

$$\mathbf{U}^* = \mathbf{U}_\mathrm{D}^* + \bar{\mathbf{U}}_\mathrm{D}^* \tag{4-47}$$

where $\bar{\mathbf{U}}_\mathrm{D}^*$ corresponds to U_{mn}^* for m and n referring to different nodes and zero values for the coefficients about the main diagonal; and $\mathbf{U}_\mathrm{D}^*$ a block-diagonal matrix with a series of two by two submatrices constituted by the still unevaluated coefficients of $\mathbf{U}^*$. Then, from Eqs. (4-46) and (4-47),

$$\mathbf{U}_\mathrm{D}^*\,\mathbf{V} = -\mathbf{W}\,\mathbf{C}\,\mathbf{V} - \bar{\mathbf{U}}_\mathrm{D}^*\,\mathbf{V} \equiv \mathbf{Y}_0 \tag{4-48}$$

which leads to a series of n_n two by two equations for each node i of the boundary:

$$\begin{bmatrix} U^*_{D(rr)} & U^*_{D(zr)} \\ U^*_{D(rz)} & U^*_{D(zz)} \end{bmatrix}_i \begin{pmatrix} V_{(r)} \\ V_{(z)} \end{pmatrix}_i = \begin{pmatrix} Y_{0(r)} \\ Y_{0(z)} \end{pmatrix}_i \qquad i = 1, ..., n_n \tag{4-49}$$

For each node on the boundary, the above equation yields two equations to solve for the four unknowns of $\mathbf{U}^*_\mathrm{D}$. Then, differently from the cases of two and three-dimensional problems of potential and elasticity in which symmetries are not included, in the present case of axisymmetry the orthogonality Eq. (4-46) is not sufficient to evaluate all undetermined coefficients of $\mathbf{U}^*_\mathrm{D}$. It is worth noticing that there is no mechanical justification to assume that $U^*_{D(zr)} = U^*_{D(rz)}$ in Eq. (4-49).

On the other hand, additional equations can be obtained by applying Eq. (4-19) to check for some of the simplest axisymmetric analytical solutions possible, exactly as in the displacement finite element method, when one requires that a formulation passes a patch test. Let $\mathbf{U} = [U_m] \in \mathbb{R}^{u_u}$ and $\mathbf{P} = [P_m] \in \mathbb{R}^{u_u}$ be displacements and nodal forces corresponding to a known analytical solution. Then, a vector of forces $\mathbf{P}^* = [P^*_m] \in \mathbb{R}^{u_u}$ of the internal reference system can be evaluated from Eq. (4-14) as

$$\mathbf{P}^* = (\mathbf{H}^\mathrm{T})^{(-1)}\,\mathbf{P} \tag{4-50}$$

in which inverse $(\mathbf{H}^\mathrm{T})^{(-1)}$ stands for a generaliszed inverse [75], since $\mathbf{H}$ is singular, as dealt with in Section 4.1.7.1. Now, applying the solution $(\mathbf{U}, \mathbf{P}^*)$ to Eq. (4-19) yields

$$(\mathbf{U}^* + \mathbf{W}\,\mathbf{C})\,\mathbf{P}^* = \mathbf{U} \tag{4-51}$$

Finally, the coefficients of $\mathbf{U}^*_\mathrm{D}$ may be expressed as

$$\mathbf{U}^*_\mathrm{D}\,\mathbf{P}^* = -\mathbf{W}\,\mathbf{C}\,\mathbf{P}^* - \bar{\mathbf{U}}^*_\mathrm{D}\,\mathbf{P}^* \equiv \mathbf{Y} \tag{4-52}$$

which can be written as a set of n_n uncoupled systems of equation for each node i of the boundary:

$$\begin{bmatrix} U^*_{D(rr)} & U^*_{D(zr)} \\ U^*_{D(rz)} & U^*_{D(zz)} \end{bmatrix}_i \begin{pmatrix} P^*_{(r)} \\ P^*_{(z)} \end{pmatrix}_i = \begin{pmatrix} Y_{(r)} \\ Y_{(z)} \end{pmatrix}_i \qquad i = 1, ..., n_n \tag{4-53}$$

Combining Eqs. (4-49) and (4-53), one obtains the following system of equations,

$$\begin{bmatrix} V_{(r)} & V_{(z)} \\ P^*_{(r)} & P^*_{(z)} \end{bmatrix}_i \left(\begin{array}{c|c} U^*_{D(rr)} & U^*_{D(rz)} \\ U^*_{D(zr)} & U^*_{D(zz)} \end{array}\right)_i = \left(\begin{array}{c|c} Y_{0(r)} & Y_{0(z)} \\ Y_{(r)} & Y_{(z)} \end{array}\right)_i \qquad i = 1, ..., n_n \tag{4-54}$$

which can be solved for the submatrices of $\mathbf{U}^*_\mathrm{D}$, provided that the expressions from

Eqs. (4-46) and (4-51) are linearly independent for all boundary nodes (that is, the matrix of system above is non-singular for all i). Since this cannot be guaranteed for any geometric configuration of the boundary, a first improvement is to resort to one more simple analytical solution (they are given in Section 4.1.5.3), with a superabundant number of equations at each node, thus solving for $\mathbf{U}^*_\mathbf{D}$ in terms of least squares.

However, as it came out in the numerical tests run with the present implementation and has already been observed in several applications of the hybrid boundary element method [59], there are some specific topological configurations for which, at some given nodes, either $V_{(r)}$ or $V_{(z)}$ in Eq. (4-49) is equal to zero (within some discretization error), and the corresponding value of $P^*_{(r)}$ or $P^*_{(z)}$ in Eq. (4-49) is also approximately zero, no matter which analytical solution one resorts to. This fact had astonished the researchers involved with the hybrid boundary element method for many years, until a sound mechanical justification for the problem could be identified and means to circumvent the difficulties satisfactorily developed [60]. In fact, the procedure just outlined is only generally applicable to an elastic body with strictly convex geometry [43].

For three-dimensional axisymmetric elasticity, the above described behavior is identified for four types of topological configuration, as shown in Fig. 4.2. If the volume of the solid is convex (and non-symmetric with reference to a horizontal plane), as in Fig. 4.2–a, $V_m \neq 0$ for all m and the procedure just outlined can be used. However, in all other cases shown in Fig. 4.2 there are nodes on the boundary where $V_m \approx 0$ (sometimes, $V_m = 0$).

If the body is non-convex with either simply-connected (Fig. 4.2–b) or multiply-connected surface (in Fig. 4.2–b, V_m approaches zero along a sequence of boundary nodes. On the other hand, if the volume is non-convex with disconnected surfaces, as in Fig. 4.2–d, $V_m = 0$ on the internal, disconnected parts of the boundary. The last case, Fig. 4.2–e, refers to the existence of symmetry of the discretized boundary, with $V_m = 0$ at the node(s) on the axis of axisymmetry even for convex volumes. The latter example is in fact the case of a not strictly convex geometry, as it also happens in the numerical simulation of a continuum with non-homogeneous properties [59, 84].

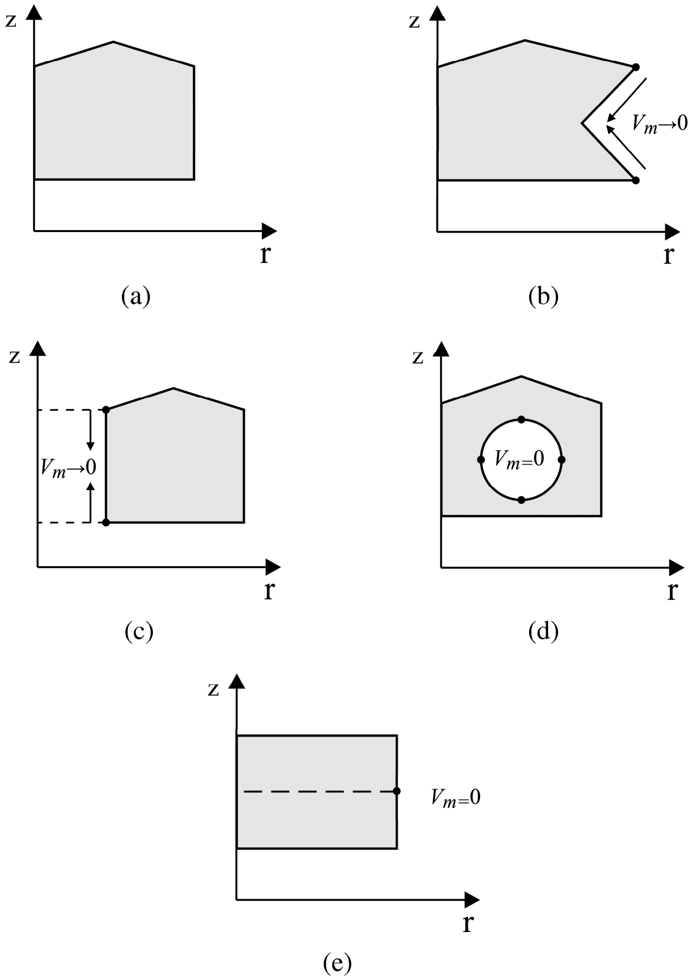


Figure 4.2: Meridian planes of an elastic axisymmetric: a) convex volume; b) non-convex volume with simply-connected surface; c) non-convex volume with multiply-connected surface; d) non-convex volume with disconnected surfaces; e) convex volume with a horizontal plane of symmetry.

As already mentioned, similar cases have been identified for potential and two-dimensional elasticity problems. In the case of a non-convex volume with disconnected surfaces, the coefficients of the submatrices of $\mathbf{U}^*_{\mathrm{D}}$ can be obtained by dealing with each disconnected surface separately and then composing the whole problem by superposition of effects. In some cases, the unknown values of $\mathbf{U}^*_{\mathrm{D}}$ corresponding to (approximately) zero values of V_m can be evaluated by interpolating the adjacent coefficients of $\mathbf{U}^*$. Although successful for some cases, such are ad hoc procedures and there is no evidence of their general applicability [52]. In fact, as exhaustively investigated, the procedure outlined above cannot be applied to the topologically demanding subject of the present work.

Fortunately, the challenging problem posed by the cases depicted in Fig. 4.2 is satisfactorily solved by means of a novel mechanical concept introduced in the next Section, as shown in the numerical examples of Chapter 6.

4.1.5
Evaluation of the coefficients about the main diagonal of $\mathbf{U}^*$ using a hybrid contragradient theorem

Equations (4-14) and (4-21) constitute the governing equations of the original version of the simplified-hybrid boundary element method [52]. An additional governing equation that may successfully substitute for (4-21) can be derived by applying a contragradient theorem, as detailed in the following.

4.1.5.1
A hybrid virtual work statement

As introduced in Section 4.1.2, $\mathbf{u}$ and $\mathbf{p}^*$ are the primary unknowns of the problem, to which correspond equivalent nodal forces and displacements $\mathbf{p}$ and $\mathbf{u}^*$, respectively. At present, one is attempting to find an expression that directly interrelates the equivalent nodal quantities $\mathbf{p}$ and $\mathbf{u}^*$. For this sake, let an admissible, virtual stress state be represented by $\mathbf{P}_\mathrm{V}^\perp \delta\mathbf{p}^*$ and $\mathbf{P}_\mathrm{W}^\perp \delta\mathbf{u}$, which are interrelated by Eq. (4-21). This virtual state is admissible because the contribution of unbalanced forces $\mathbf{P}_\mathrm{V}\delta\mathbf{p}^*$ and rigid body displacements $\mathbf{P}_\mathrm{W}\delta\mathbf{u}$ is excluded. For consistency, $\delta\mathbf{u} = 0$ on Γ_u, but the inclusion of boundary conditions may be postponed. The virtual energy $\delta U(\mathbf{u})$ related to the pair $(\mathbf{P}_\mathrm{W}^\perp \delta\mathbf{u}, \mathbf{p})$ and the complementary virtual energy $\delta U^C(\mathbf{p}^*)$ related to the pair $(\mathbf{u}^*, \mathbf{P}_\mathrm{V}^\perp \delta\mathbf{p}^*)$, are given by

$$\delta U(\mathbf{u}) = \delta\mathbf{u}^\mathrm{T}\mathbf{P}_\mathrm{W}^\perp\,\mathbf{p} \tag{4-55}$$

$$\delta U^C(\mathbf{p}^*) = \delta\mathbf{p}^{*\mathrm{T}}\mathbf{P}_\mathrm{V}^\perp\,\mathbf{u}^* \tag{4-56}$$

For linear elastic deformation, $\delta U(\mathbf{u})$ and $\delta U^C(\mathbf{p}^*)$ are equivalent. Then,

$$\delta\mathbf{u}^\mathrm{T}\mathbf{P}_\mathrm{W}^\perp\,\mathbf{p} = \delta\mathbf{p}^{*\mathrm{T}}\mathbf{P}_\mathrm{V}^\perp\,\mathbf{u}^* \tag{4-57}$$

Substituting for $\delta\mathbf{u}^\mathrm{T}\mathbf{P}_\mathrm{W}^\perp$ according to Eq. (4-21), one obtains

$$\delta\mathbf{p}^{*\mathrm{T}}\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_W^\perp\,\mathbf{p} = \delta\mathbf{p}^{*\mathrm{T}}\mathbf{P}_V^\perp\mathbf{u}^* \tag{4-58}$$

an expression that is valid for any virtual set of point forces $\delta\mathbf{p}^*$ (not just the admissible ones), thus resulting

$$\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_\mathrm{W}^\perp\,\mathbf{p} = \mathbf{P}_\mathrm{V}^\perp\mathbf{u}^* \tag{4-59}$$

This might be the final expression one is looking for. However, resorting to another contragradient expression, Eq. (4-18), and also to Eq. (4-25), it is possible to arrive at a more convenient equation,

$$\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_{\mathrm{W}}^{\perp}\,\mathbf{p} = \mathbf{H}\,\mathbf{u} \tag{4-60}$$

that explicitly relates nodal displacements and equivalent nodal forces without the intervenience of the auxiliary set of forces $\mathbf{p}^*$, as originally proposed in terms of Eqs. (4-14) and (4-21). The construction of a stiffness relation from Eq. (4-60) is addressed in Section 4.1.6.

4.1.5.2
A contragradient nodal displacement compatibility statement

The derivation of a new governing equation, as presented in Section 4.1.5.1 by applying a hybrid contragradient theorem, was motivated by the need of an equation that allowed the evaluation of the unknown coefficients $\mathbf{U}_{\mathrm{D}}$ of $\mathbf{U}^*$, as laid down in Eq. (4-47), regardless topological issues.

As in the previous section, let $\mathbf{U}$ and $\mathbf{P}$ be the displacement and equivalent nodal force vectors that correspond to some simple analytical solution. Applying this solution to Eq. (4-60), one obtains

$$\mathbf{U}^{*\mathrm{T}}\,\mathbf{P} = \mathbf{H}\,\mathbf{U} \tag{4-61}$$

since $\mathbf{P}_{\mathbf{W}}^{\perp}\,\mathbf{P} = \mathbf{0}$, according to Eq. (4-30). It is worth noticing that this equation is contragradient to Eq. (4-51), thus establishing that the evaluation of equivalent nodal displacements $\mathbf{u}^*$ should lead to the same result whether starting from equivalent nodal forces $\mathbf{P}$ or from nodal displacements $\mathbf{u}$. The use of Eq. (4-61) has two decisive advantages as compared with a procedure based on Eq. (4-51): (1) a simple analytical solution expressed in terms of $\mathbf{P}$ does not depend on the problem's topology, as it occurs with $\mathbf{P}^*$; (2) most importantly, the evaluation of $\mathbf{P}^*$ – via the solution of Eq. (4-14) – is no longer necessary.

The matrix $\mathbf{U}^*$ can be split as in Eq. (4-47), so that Eq. (4-61) becomes

$$\mathbf{U}_{\mathrm{D}}^{*\mathrm{T}}\,\mathbf{P} = \mathbf{H}\,\mathbf{U} - \bar{\mathbf{U}}_{\mathrm{D}}^{*\mathrm{T}}\,\mathbf{P} \equiv \mathbf{Y} \tag{4-62}$$

which corresponds to uncoupled sets of equation for each node i of the boundary,

$$\begin{bmatrix} U^*_{D(rr)} & U^*_{D(zr)} \\ U^*_{D(rz)} & U^*_{D(zz)} \end{bmatrix}_i \begin{pmatrix} P_{(r)} \\ P_{(z)} \end{pmatrix}_i = \begin{pmatrix} Y_{(r)} \\ Y_{(z)} \end{pmatrix}_i \qquad i = 1, ..., n_n \tag{4-63}$$

Each analytical solution of the axisymmetric problem provides two equations. Therefore, at least two linearly independent analytical solutions are necessary for the evaluation of the four unknowns $\mathbf{U}^*_{\mathrm{D}}$ related to node i. In general, three analytical solutions are required to ensure a solution for any given boundary geometry, namely $(\mathbf{U_1}, \mathbf{P_1})$, $(\mathbf{U_2}, \mathbf{P_2})$ and $(\mathbf{U_3}, \mathbf{P_3})$, to be presented in the next Section. However, in the case of a node i on the axis of axisymmetry $\xi = 0$, an additional analytical solution, namely $(\mathbf{U_4}, \mathbf{P_4}) = (\mathbf{A}_i, \mathbf{0})$, must be included. This solution, where $\mathbf{A_i}$ is the i_{th} column of $\mathbf{A}$, expresses, according to Eq. (4-39), that radial ring loads applied on the axis of axisymmetry produce no displacements.

The analytical solutions can be applied to each node i of the boundary to obtain

$$\left[\begin{array}{cc} P_{1\,(r)} & P_{1\,(z)} \\ P_{2\,(r)} & P_{2\,(z)} \\ P_{3\,(r)} & P_{3\,(z)} \\ \hline 1 & 0 \end{array}\right]_i \left(\begin{array}{c|c} U^*_{D\,(rr)} & U^*_{D\,(rz)} \\ U^*_{D\,(zr)} & U^*_{D\,(zz)} \end{array}\right)_i = \left(\begin{array}{c|c} Y_{1\,(r)} & Y_{1\,(z)} \\ Y_{2\,(r)} & Y_{2\,(z)} \\ Y_{3\,(r)} & Y_{3\,(z)} \\ \hline 0 & 0 \end{array}\right)_i \qquad (4\text{-}64)$$

where the fourth row of equations is to be included only if $\xi = 0$. This complementary row follows from Eq. (4-34). Finally, the unknown coefficients of $\mathbf{U}^*$ are evaluated by solving the above equation in terms of least squares for each node i.

As mentioned before, $\mathbf{U}^*$ is symmetric for the coefficients referring to distinct nodes. However, there is no theoretical reason for $U^*_{(rz)} = U^*_{(rz)}$ when these coefficients refer to the same nodes, since its evaluation depends on the overall discretization of the body. Several numerical examples demonstrated that the results are more accurate when the symmetry is not imposed in the least squares system of equations, as adopted in this work. As the discretization of the boundary is refined, $\mathbf{U}^*$ tends to symmetry and the accuracy of the procedure can be assessed by means of Eq. (4-32), for instance.

The main advantage of this procedure is its validity for any topological configuration, since it does not depend on $\mathbf{V}$. Also, the explicit evaluation of $\mathbf{C}$ is not necessary in this formulation. The above procedure can be applied directly to non-convex volumes with disconnected surfaces, with no need of further developments in terms of complementary domain.

The axisymmetric analytical solutions referred to in this section are presented in the following Section.

4.1.5.3
Axisymmetric analytical solutions

To ensure that each node of the boundary has at least two linearly independent analytical solutions, no matter its coordinate direction (r, z) and outward normal (η_r, η_z), at least three analytical solutions are necessary. These functions were derived as the simplest non-singular solutions of the Navier-Cauchy equilibrium equations, given by Eqs. (2-26) and (2-28), also ensuring that they orthogonal to each other. Moreover, these analytical solutions are valid for any value of ν, then also for incompressible solids.

Analytical solution 1

The simplest solution that satisfies

$$\sigma_{rr} = 1, \quad \sigma_{rz} = 0 \quad \text{and} \quad \sigma_{zz} = 0 \tag{4-65}$$

is

$$\mathbf{u_1} = \begin{pmatrix} \frac{r\,(1-\nu)}{2\mu\,(1+\nu)} \\ -\frac{z\,\nu}{\mu\,(1+\nu)} \end{pmatrix} \quad \text{and} \quad \mathbf{t_1} = \begin{pmatrix} \eta_r \\ 0 \end{pmatrix} \tag{4-66}$$

Analytical solution 2

The simplest solution that satisfies

$$\sigma_{rr} = 0, \quad \sigma_{rz} = 0 \quad \text{and} \quad \sigma_{zz} = 1 \tag{4-67}$$

is

$$\mathbf{u_2} = \begin{pmatrix} -\frac{r\,\nu}{2\mu\,(1+\nu)} \\ \frac{z}{2\mu\,(1+\nu)} \end{pmatrix} \quad \text{and} \quad \mathbf{t_2} = \begin{pmatrix} 0 \\ \eta_z \end{pmatrix} \tag{4-68}$$

Analytical solution 3

The simplest non-singular stress state, orthogonal to the analytical solutions 1 and 2, is in principle

$$\sigma_{rr} = 0, \quad \sigma_{rz} = -r/2 \quad \text{and} \quad \sigma_{zz} = z \tag{4-69}$$

with corresponding displacements and traction forces

$$\tilde{\mathbf{u}}_3 = \begin{pmatrix} -\frac{r\,z\,\nu}{2\mu\,(1+\nu)} \\ \frac{z^2-r^2}{4\mu\,(1+\nu)} \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{t}}_3 = \begin{pmatrix} -\frac{r\,\eta_z}{2} \\ -\frac{r\,\eta_r}{2} + z\,\eta_z \end{pmatrix} \tag{4-70}$$

The fundamental solutions do not depend on the nodal coordinates but only on the relative positions between the source and field points. Accordingly, the matrices $\mathbf{U}^*$ and $\mathbf{H}$ do not change if the coordinate system is translated in the direction z, and the same is expected with respect to the unknown coefficients about the main diagonal of $\mathbf{U}^*$. For the analytical solutions 1 and 2, a translation c_z of coordinate does not change the traction force values and only gives rise to rigid body displacements,

$$\mathbf{u}_1^c = \begin{pmatrix} 0 \\ -\frac{c_z\,\nu}{\mu\,(1+\nu)} \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2^c = \begin{pmatrix} 0 \\ \frac{c_z}{2\mu\,(1+\nu)} \end{pmatrix} \tag{4-71}$$

which are filtered when multiplied by $\mathbf{H}$, according to Eq. (4-24). As a consequence, a coordinate translation c_z does not interfere in the evaluation of the unknown values of $\mathbf{U}^*$, if only the analytical solutions 1 and 2 are taken into account. On the other hand, the non-linear term z^2 of $\tilde{t}_{3z}$ in Eq. (4-69) leads to a change of the constant traction forces for a coordinate translation c_z:

$$\tilde{\mathbf{u}}_3^c = \begin{pmatrix} -\frac{c_z\,r\,\nu}{2\mu\,(1+\nu)} \\ \frac{2z\,c_z+c_z^2}{4\mu\,(1+\nu)} \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{t}}_3^c = \begin{pmatrix} 0 \\ c_z\,\eta_z \end{pmatrix} \tag{4-72}$$

Then the analytical solution 3 is best referred to a node i of coordinates (r_i, z_i), in order to become invariant to coordinate translation,

$$\mathbf{u}_3 = \begin{pmatrix} -\frac{r\,(z-z_i)\,\nu}{2\mu\,(1+\nu)} \\ \frac{(z-z_i)^2-r^2}{4\mu\,(1+\nu)} \end{pmatrix} \quad \text{and} \quad \mathbf{t}_3 = \begin{pmatrix} -\frac{r\,\eta_z}{2} \\ -\frac{r\,\eta_r}{2} + (z - z_i)\,\eta_z \end{pmatrix} \qquad i = 1, .., n_n \tag{4-73}$$

or in terms of Eqs. (4-68) and (4-70), to simplify the numerical calculation, for each node i

$$\mathbf{u}_3 = \tilde{\mathbf{u}}_3 - z_i\,\mathbf{u}_2 \tag{4-74}$$

$$\mathbf{t}_3 = \tilde{\mathbf{t}}_3 - z_i\,\mathbf{t}_2 \tag{4-75}$$

4.1.6
Stiffness matrix

In the original version of the simplified-hybrid boundary element method, the stiffness matrix was obtained by solving for $\mathbf{p}^*$ in Eqs. (4-14) and Eq. (4-21), resulting in the system expressed by Eq. (3-14). The corresponding stiffness matrix is given by

$$\mathbf{K} = \mathbf{H}^{\mathrm{T}}\,(\mathbf{P}_{\mathrm{W}}^{\perp}\,\mathbf{U}^*)^{(-1)}\,\mathbf{P}_{\mathrm{W}}{}^{\perp} \tag{4-76}$$

An alternative expression for the stiffness matrix can be obtained by just solving for $\mathbf{p}$ in Eq. (4-60):

$$\mathbf{K} = (\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_{\mathrm{W}}^{\perp})^{(-1)}\,\mathbf{H} \tag{4-77}$$

The generalized inverses [75] of the singular matrices $\mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^*$ and $\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_\mathrm{W}^\perp$ are discussed in the next Section. These inverses are such that

$$(\mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^*)^{(-1)} = \left[(\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_\mathrm{W}^\perp)^{(-1)}\right]^\mathrm{T} \tag{4-78}$$

This work adopts the stiffness matrix given by Eq. (4-77) since it uses the transposed $\mathbf{U}^*$, for which their unknown coefficients are evaluated in the procedure presented in Section 4.1.5.2. It may be shown that with increasing mesh refining the expressions of Eqs. (4-76) and (4-77) tend to be equivalent and $\mathbf{K}$ tends to become symmetric.

In the following, two alternative ways of computing the generalized inverses of the two above equations are outlined. Of course, the procedure of Section 4.1.6.2 is the adequate one for implementation purposes, as it does not make use of the orthogonal basis $\mathbf{V}$. In fact, the outline of Section 4.1.6.1 is given only for historical reasons.

4.1.6.1 Generalized inverses of $\mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^*$

The matrix $\mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^*$ is singular and its rank depends on the number of boundary nodes on the axis of axisymmetry. In this case, all the coefficients of the rows corresponding to the r-direction of $\mathbf{U}^*$ are void, since $u^{*c}_{i(r)} = 0$ for $\xi = 0$. Thus, one must distinguish the following cases.

Case 1: $\Gamma(r, z)$ does not intercept the axis of axisymmetry

In this case, rank $\mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^* = n_u - 1$, where n_u is the number of displacement degrees of freedom.

The inverse of $\mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^*$ can be obtained in the frame of the Bott-Duffin inverse [76, 75] for the solution of $\mathbf{p}^*$ in the following restricted system

$$\begin{cases} \mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^*\,\mathbf{p}^* = \mathbf{P}_\mathrm{W}^\perp\mathbf{u} \\ \mathbf{P}_\mathrm{V}\,\mathbf{p}^* = \mathbf{0} \end{cases} \tag{4-79}$$

which refers to the transformation of forces $\mathbf{p}^*$ into displacements $\mathbf{P}_\mathrm{W}^\perp\mathbf{u}$ in the space spanned by $\mathbf{P}_\mathrm{V}^\perp$. Since matrix $\mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^*$ does not transform unbalanced forces of the space spanned by $\mathbf{P}_\mathrm{V}$, as shown in Eq. (4-32), its inverse can be written as

$$(\mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^*)^{(-1)} = \mathbf{P}_\mathrm{V}^\perp\,(\mathbf{P}_\mathrm{W}^\perp\,\mathbf{U}^*\mathbf{P}_\mathrm{V}^\perp + \lambda\,\mathbf{P}_\mathrm{V})^{-1} \tag{4-80}$$

The term in brackets is by construction non-singular and λ is a constant of order $1/\mu$ to make sure that the summands have approximately the same order of magnitude.

Case 2: $\Gamma(r, z)$ intercepts the axis of axisymmetry

Let n_a be the number of points at which $\Gamma(r, z)$ intersects the axis of axisymmetry. In this case, rank $\mathbf{P}_\mathrm{W}^\perp \mathbf{U}^* = n_u - n_a - 1$. The number n_a refers to the radial ring sources applied on the axis of axisymmetry of basis $\mathbf{A}$, given by Eq. (4-34) and introduced in Section 3.1.4.

The corresponding orthogonal and complementary orthogonal projectors are

$$\mathbf{P}_\mathrm{AV} = \mathbf{P}_\mathrm{A} + \mathbf{P}_\mathrm{V} \tag{4-81}$$

$$\mathbf{P}_\mathrm{AV}^\perp = \mathbf{I} - \mathbf{P}_\mathrm{AV} \tag{4-82}$$

Similarly to the inverse presented in Case 1 and considering Eqs. (4-39) and (4-32), the inverse of $\mathbf{U}^{*\mathrm{T}} \mathbf{P}_\mathrm{W}^\perp$ is expressed by

$$(\mathbf{P}_\mathrm{W}^\perp \mathbf{U}^*)^{(-1)} = \mathbf{P}_\mathrm{AV}^\perp (\mathbf{P}_\mathrm{W}^\perp \mathbf{U}^* \mathbf{P}_\mathrm{AV}^\perp + \lambda \mathbf{P}_\mathrm{AV})^{-1} \tag{4-83}$$

4.1.6.2 Generalized inverses of $\mathbf{U}^{*\mathrm{T}} \mathbf{P}_\mathrm{W}^\perp$

Simillarly to the inverse of $\mathbf{P}_\mathrm{W}^\perp \mathbf{U}^*$, one must distinguish the following two inverses.

Case 1: $\Gamma(r, z)$ does not intercept the axis of axisymmetry

In this case, rank $\mathbf{U}^{*\mathrm{T}} \mathbf{P}_\mathrm{W}^\perp = n_u - 1$.

The inverse of $\mathbf{U}^{*\mathrm{T}} \mathbf{P}_\mathrm{W}^\perp$ can be obtained in the frame of the Bott-Duffin inverse [76, 75] for the solution of $\mathbf{p}$ in the following restricted system

$$\begin{cases} \mathbf{U}^{*\mathrm{T}} \mathbf{P}_\mathrm{W}^\perp \mathbf{p} = \mathbf{H}\mathbf{u} \\ \mathbf{P}_\mathrm{W} \mathbf{p} = \mathbf{0} \end{cases} \tag{4-84}$$

which refers to the transformation of equivalent forces $\mathbf{p}$ into displacements $\mathbf{H}\mathbf{u}$ in the space spanned by $\mathbf{P}_\mathbf{W}^\perp$. Since matrix $\mathbf{U}^{*\mathrm{T}} \mathbf{P}_\mathrm{W}^\perp$ does not transform unbalanced forces of the space spanned by $\mathbf{P}_\mathrm{W}$, as shown in Eq. (4-33), its inverse can be written as

$$(\mathbf{U}^{*\mathrm{T}} \mathbf{P}_\mathrm{W}^\perp)^{(-1)} = \mathbf{P}_\mathrm{W}^\perp (\mathbf{U}^{*\mathrm{T}} \mathbf{P}_\mathrm{W}^\perp + \lambda \mathbf{P}_\mathbf{W})^{-1} \tag{4-85}$$

in which the term in brackets is by construction non-singular.

Case 2: $\Gamma(r, z)$ intercepts the axis of axisymmetry

In this case, rank $\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_{\mathrm{W}}^{\perp} = n_u - n_a - 1$. The number n_a refers to the radial ring sources applied on the axis of axisymmetry of basis **A**, given by Eq. (4-34) and introduced in Section 3.1.4.

The corresponding orthogonal and complementary orthogonal projectors are

$$\mathbf{P}_{\mathrm{AW}} = \mathbf{P}_{\mathrm{A}} + \mathbf{P}_{\mathrm{W}} \tag{4-86}$$

$$\mathbf{P}_{\mathrm{AW}}^{\perp} = \mathbf{I} - \mathbf{P}_{\mathrm{AW}} \tag{4-87}$$

Similarly to the inverse presented in Case 1, the inverse of $\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_{\mathrm{W}}^{\perp}$ is expressed by

$$(\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_{\mathrm{W}}^{\perp})^{(-1)} = \mathbf{P}_{\mathrm{AW}}^{\perp}\,(\mathbf{U}^{*\mathrm{T}}\,\mathbf{P}_{\mathrm{W}}^{\perp} + \lambda\,\mathbf{P}_{\mathrm{AW}})^{-1} \tag{4-88}$$

For the Case 2 of the inverses discussed in this section, the resulting stiffness matrix **K** has its rank reduced by -1 for each node on the axis of axisymmetry. In this case, prescription of zero radial displacements at these nodes provides the additional conditions for the system of equations to be solved.

4.1.7
Displacements and stresses in the domain

The displacements u_i^* and the stresses σ_{ij}^* at a point $\mathrm{Q}(r, z)$ in the domain Ω can be evaluated by Eqs. (4-3), (4-2) and (4-4). The rigid body displacement constant $C_m\, p_m^* \equiv \mathbf{C}\,\mathbf{p}^*$ is given in terms of nodal displacements along the boundary by Eq. (4-20). The forces $\mathbf{p}^*$ can be solved in Eq. (4-14) in terms of equivalent nodal forces,

$$\mathbf{p}^* = (\mathbf{H}^{\mathrm{T}})^{(-1)}\,\mathbf{p} \tag{4-89}$$

In the above equation, since the matrix **H** is singular, the inverse $(\mathbf{H}^{\mathrm{T}})^{(-1)}$ must be obtained in the frame of generalized inverses [75], as presented in the following.

4.1.7.1
Generalized inverse of $\mathbf{H}^{\mathrm{T}}$

Consider the unbalanced forces of basis **V** that produce rigid body displacements in the z-direction and the transformation performed by the matrix $\mathbf{H}^{\mathrm{T}}$ in Eq. (4-14). Its inverse can be obtained as a particular case of the Bott-Duffin inverse [76, 75] for the solution of $\mathbf{p}^*$ in the following restricted system

$$\begin{cases} \mathbf{H}^{\mathrm{T}}\,\mathbf{p}^* = \mathbf{p} \\ \mathbf{P}_{\mathrm{V}}\,\mathbf{p}^* = \mathbf{0} \end{cases} \tag{4-90}$$

which refers to the transformation of auxiliary forces $\mathbf{p}^*$ of the space spanned by $\mathbf{P}_{\mathrm{V}}^{\perp}$ into forces $\mathbf{p}$. Since $\mathbf{H}^{\mathrm{T}}$ does not transform unbalanced forces of the space spanned by $\mathbf{P}_{\mathrm{V}}$, as expressed in Eq. (4-25), its inverse can be written as

$$(\mathbf{H}^{\mathrm{T}})^{(-1)} = \mathbf{P}_{\mathrm{V}}^{\perp}\,(\mathbf{H}^{\mathrm{T}} + \mathbf{P}_{\mathrm{V}})^{-1} \tag{4-91}$$

in which the term in brackets is by construction a non-singular term.

The procedure outlined above requires the evaluation of $\mathbf{V}$ and $\mathbf{p}^*$, which involves the inversion of two matrices in Eqs. (4-91) and (4-29). There are alternative ways of finding the generalized inverse of $\mathbf{H}$ that circumvent the evaluation of the null space $\mathbf{V}$, as in terms of least squares $\mathbf{H}^{(-1)} = \mathbf{P}_{\mathrm{W}}^{\perp}\left(\mathbf{H}^{\mathrm{T}}\mathbf{H} + \mathbf{P}_{\mathrm{W}}\right)^{-1}\mathbf{H}^{\mathrm{T}}$ [84]. However, the displacements and stresses in the domain can be assessed without any additional integration also in the case of points that are close to the boundary. This aspect is specially advantageous for axisymmetric problems, since, as outlined in Section 3.2.3 for the conventional boundary element method, the usual way of evaluating results at internal points requires the integration of long and complex functions. The steps presented in the next Section can also be applied to the conventional boundary element method, as presented by Gaul et al. [85] for problems of elasticity and diffusion in the frequency domain.

4.1.8
Stresses on the boundary

The stresses σ_{ij} at a point $\mathrm{Q}(r, z)$ on the boundary can be obtained by considering that the approximation for stresses σ^*_{ij} in the domain Ω is also valid on the boundary Γ. Applying Eq. (4-2) to each node of the boundary, one obtains

$$\sigma_{np} = \Sigma^*_{npm}\, p^*_m \quad \text{or} \quad \sigma = \mathbf{\Sigma}^*\,\mathbf{p}^* \tag{4-92}$$

in which the auxiliary nodal forces $\mathbf{p}^*$ are calculated by Eq. (4-89) from known equivalent nodal forces [42]. The index m refers to both the direction and the point of application $\mathrm{P}(\xi, z')$ of the ring load. The indices n and p denote the stress component σ_{np} to be evaluated at point $\mathrm{Q}(r, z)$.

When the points $\mathrm{P}(\xi, z')$ and $\mathrm{Q}(r, z)$ coincide, $\rho = 0$ and the functions $\sigma^{*c}_{ijm}(\mathrm{Q}, \mathrm{P})$ become singular. Therefore, analogously to the case of matrix $\mathbf{U}^*$ presented in Sections 4.1.4 and 4.1.5, some coefficients of Σ^*_{npm} cannot be directly evaluated.

Let $\mathbf{\Sigma}$ and $\mathbf{P}^*$ be the nodal stresses and auxiliary nodal forces provided by an axisymmetric analytical solution. Also, let the tensor $\mathbf{\Sigma}^*$ be split into

$$\mathbf{\Sigma}^* = \mathbf{\Sigma}^*_{\mathbf{D}} + \bar{\mathbf{\Sigma}}^*_{\mathbf{D}} \tag{4-93}$$

in which $\mathbf{\Sigma}^*_{\mathbf{D}}$ is constituted by the unknown coefficients of $\mathbf{\Sigma}^*$; and $\bar{\mathbf{\Sigma}}^*_{\mathbf{D}}$ is completely known. Applying Eq. (4-92) iyelds

$$\mathbf{\Sigma}^*_{\mathbf{D}}\,\mathbf{P}^* = \mathbf{\Sigma} - \bar{\mathbf{\Sigma}}^*_{\mathbf{D}}\,\mathbf{P}^* \equiv \mathbf{Y} \tag{4-94}$$

that can be rewritten for each node as

$$\begin{bmatrix} \Sigma^*_{D\,(rr)\,(r)} & \Sigma^*_{D\,(rr)\,(z)} \\ \Sigma^*_{D\,(rz)\,(r)} & \Sigma^*_{D\,(rz)\,(z)} \\ \Sigma^*_{D\,(zz)\,(r)} & \Sigma^*_{D\,(zz)\,(z)} \end{bmatrix}_i \begin{pmatrix} P^*_{(r)} \\ P^*_{(z)} \end{pmatrix}_i = \begin{pmatrix} Y_{(rr)} \\ Y_{(zz)} \end{pmatrix}_i \qquad i = 1, ..., n_n \tag{4-95}$$

In the above equation, each auxiliary axisymmetric analytical solution provides three equations. Therefore, two linearly independent analytical solutions are needed in order to solve the system for the six unknown coefficients of $\mathbf{\Sigma}^*_{\mathbf{D}}$ for each node. Three auxiliary analytical solutions $(\mathbf{\Sigma_1}, \mathbf{P}^*_{\mathbf{1}})$, $(\mathbf{\Sigma_2}, \mathbf{P}^*_{\mathbf{2}})$ e $(\mathbf{\Sigma_3}, \mathbf{P}^*_{\mathbf{3}})$ were presented in Section 4.1.5.3. One may also employ the orthogonality relation $\mathbf{\Sigma}^*\,\mathbf{V} = \mathbf{0}$ for the unbalanced forces that produces no stresses, which constitute the auxiliary solution $(\mathbf{0}, \mathbf{V})$. If the node i is placed on the axis of axisymmetry, the solution $(\mathbf{A}_i, \mathbf{0})$ must also be taken into account.

Finally, the unknown coefficients of $\mathbf{\Sigma}^*$ can be obtained by solving the three systems of equations by least squares for each node i, similarly to the procedure presented in Section 4.1.4 for $\mathbf{U}^*$.

As mentioned in Section 4.1.4.2, $V_m = 0$ at some boundary nodes in case of a non-convex domain. This feature is consistent with the fact that the stress gradient on regions close to such nodes has a local character and, whenever relevant in terms of numerical solution, should be investigated appropriately. It is the case of stress fields close to notches or around a crack tip.

Alternatively, one may also assess the stresses along the boundary by interpolating the results in a local coordinate system, as presented in Section 3.1.7 for the conventional boundary element method, and adopted in this work.

4.2
Formulation for the axisymmetric halfspace problem

Figure 3.5 illustrates an axisymmetric problem in the halfspace. The developments for the axisymmetric halfspace problem – in the context of the simplified-hybrid boundary element method – follow the same steps of the formulation for the fullspace problem, as presented in Section 4.1, except that the fundamental solutions are given as in Section 2.3 and, as a result, the rigid body translation in the z direction is precluded. Then, the whole development of Section 4.1 is formally applicable by simply removing any references to the inadmissible spaces spanned by **W** and **V**. Therefore, this particular formulation is not repeated in the present Section.

In the case of a halfspace problem, any additional numerical difficulties are in principle only related to the evaluation of the matrix **H**, which has already been dealt with in Chapter 3. Moreover, no formal evaluation of generalized inverses is required, as the only inadmissible spaces refer to the cases of radial ring loads applied at nodes on the axisymmetry axis, which can be dealt with exactly as described for the Case 2 of Section 3.1.4.1.

However, too much effort has been invested in trying to solve the present problem – as well as the fullspace problem – in the frame of the procedure outlined in Section 4.1.4 in order to evaluate the coefficients of the submatrices about the main diagonal of $\mathbf{U}^*$. Time and extensive code writing have been also unduly devoted to solve the halfspace problem by splitting the fundamental solution into two parts, since only the terms referring to the fullspace problem present undetermined coefficients.

On the other hand, the procedure presented in Section 4.1.5 on the basis of a hybrid contragradient theorem makes the problem trivial – provided that one finds some simple analytical solutions for the halfspace problem to substitute for the ones outlined in Section 4.1.5.3.

Finding simple, useful analytical solutions for the halfspace problem is not an easy task. The first solution of Section 4.1.5.3 is a viable one. However, several sets of solution developed until now present the drawback referred to for Eqs. (4-69) and (4-70), namely, that the formulation becomes dependent on the nodal coordinate z_i. This would be expected and is per se not a drawback, as the formulation for the halfspace problem is intrinsically dependent on the distance of a point to the surface $z = 0$. There is still more work to be done on the mathematics of the subject, until the numerical results for the elements about the main diagonal of $\mathbf{U}^*$ become as satisfactory as in the case of the fullspace problem, for problems of any configuration.

For problems with Neumann boundary conditions, Eq. (4-14) is the only one required in the solution of the problem. Owing to time restriction, this is the only formulation implemented for halfspace problems in the frame of the simplified-hybrid boundary element method, as developed in this thesis and presented in Chapter 5. One hopes to find a solution for the problem as soon as possible in order to make the formulation applicable to any kind of problem.